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# AN EXPOSITION ON HANSEN'S METHOD OF PARTIAL ANOMALIES


GEORGE E. McCLUSKEY, JR.




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SUMMARY

Hansen's method of partial anomalies is described and his application of the method to Encke's comet is discussed. The principle innovation of the method is to divide the ellipse of the perturbed body into a small number of segments. In each segment an independent variable closely related to the conventional anomalies is selected in order that the series representing the perturbations is strongly convergent within the segment but is invalid outside of it. A different set of series is obtained for each segment of the ellipse. In the case of high eccentricity, these series can be made much more convergent than the conventional series.

This method may be applied to the determination of the perturbations of artificial satellites and space probes with highly eccentric orbits. The "IMP and AIMP" type satellites and probes to Venus, Mars, or Jupiter could be handled by this theory. In fact, any celestial object, natural or artificial, with an eccentricity greater than about 0.50 may be very efficiently dealt with by means of the method of partial anomalies. This method has the great advantage of being rapidly convergent for all eccentricities and of being semianalytic.

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HANSEN'S METHOD OF PARTIAL ANOMALIES

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## INTRODUCTION

The perturbations which one planet produces on another or on a comet depend on the mean anomalies of the two bodies. The perturbations may be expressed in the form:

$$\sum_{i}\sum_{i'} p_{i,i'} \sin\left(ig + i'g' + P_{i,i'}\right)$$

where $p_{i,i'}$ and $P_{i,i'}$ are functions of the elliptic orbital elements of the two bodies and of $i$ and $i'$ which are integers, $i$ extending from 1 to $\infty$ and $i'$ from $-\infty$ to $+\infty$. The mean anomaly of the perturbed body is $g$ while that of the perturbing body is $g'$.

If one of the bodies has an orbit of high eccentricity and is highly inclined with respect to the other, the above series will converge very slowly, if at all. In the case of a comet the radius vector may at one time be much smaller than that of the perturbing planet and at another time much larger. Thus, a series using one of the conventional anomalies as its argument may converge in one part of the orbit but will fail to do so in another part.

In an effort to correct this situation, Hansen (Reference 1) has chosen a new variable to represent the motion of the perturbed body in place of the mean anomaly. This new variable is called the partial anomaly. The partial anomaly, when applied properly, will lead to very rapidly converging series with respect to the perturbed body for any value of the eccentricity less than unity. The method may even be applied to parabolic and hyberbolic motion.

Hansen introduces two partial anomalies, one known as the inferior anomaly and the other as the superior anomaly, and also an intermediate anomaly. By means of these anomalies the ellipse is divided into two, three, or any number of segments. Each segment is represented by one particular partial anomaly. The equations of the ellipse and the equations of elliptic motion are expressed in terms of the partial anomalies, as are the perturbations. Each segment has its own set of series for the perturbations. A given set of series cannot be used outside of its particular segment but will converge rapidly within its segment.

# I. THE PARTIAL ANOMALIES

Let:

$f$ = True anomaly

$u$ = Eccentric anomaly

$k$ = Inferior partial anomaly

$k_1$ = Superior partial anomaly

$a$ = Semimajor axis

$e$ = Eccentricity

$r$ = Radius vector

$n$ = Mean motion (Kepler's third law)

$t$ = Time

Define $k$ and $k_1$ by:

$$\sin\frac{1}{2}u = \epsilon(\cos X \sin k + \sin X)$$

$$\cos\frac{1}{2}f = \epsilon_1\left(\cos X_1 \sin k_1 - \sin X_1\right) \tag{1}$$

where:

$$\epsilon = \left[\frac{r' + r'' - 2a(1-e)}{4ae}\right]^{1/2}$$

$$\epsilon_1 = \left[\frac{a(r' + r'')(1+e) - 2r'r''}{4er'r''}(1-e)\right]^{1/2}$$

$$\tan(45 - X) = \left[\frac{r'' - a(1-e)}{r' - a(1-e)}\right]^{1/2}$$

$$\tan\left(45 - X_1\right) = \left[\frac{a(1+e) - r'}{a(1+e) - r''} \cdot \frac{r''}{r'}\right]^{1/2} \tag{2}$$

$r'$ and $r''$ being any two radius vectors of the ellipse.

Equations (2) show that:

$$0 \leq \epsilon \leq 1$$

$$0 \leq \epsilon_1 \leq 1$$

$$-\frac{\pi}{4} \leq X \leq +\frac{\pi}{4}$$

$$-\frac{\pi}{4} \leq X_1 \leq +\frac{\pi}{4}$$

The equations of the ellipse are:

$$r' = a(1 - e\cos u')$$

$$r'' = a(1 - e\cos u'') \tag{3}$$

or:

$$r' = \frac{a\left(1 - e^2\right)}{1 + e\cos f'}$$

$$r'' = \frac{a\left(1 - e^2\right)}{1 + e\cos f''} \tag{4}$$

Using Equations (3) and (4), we may write Equations (2) in the following forms:

$$\epsilon = \left[\frac{\sin^2 \frac{1}{2} u'' + \sin^2 \frac{1}{2} u'}{2}\right]^{1/2}$$

$$\epsilon_1 = \left[\frac{\cos^2 \frac{1}{2} f'' + \cos^2 \frac{1}{2} f'}{2}\right]^{1/2}$$

$$\tan(45 - X) = \frac{\sin \frac{1}{2} u''}{\sin \frac{1}{2} u'}$$

$$\tan\left(45 - X_1\right) = -\frac{\cos \frac{1}{2} f'}{\cos \frac{1}{2} f''} \tag{5}$$

or:

$$\epsilon \cos X = \frac{\sin \frac{1}{2} u'' + \sin \frac{1}{2} u'}{2}$$

$$\epsilon \sin X = \frac{\sin \frac{1}{2} u' - \sin \frac{1}{2} u''}{2}$$

$$\epsilon_1 \cos X_1 = \frac{\cos\frac{1}{2} f' - \cos\frac{1}{2} f''}{2}$$

$$\epsilon_1 \sin X_1 = -\frac{\cos\frac{1}{2} f' + \cos\frac{1}{2} f''}{2} \tag{6}$$

or:

$$\sin 2X = \frac{r' - r''}{4ae\epsilon^2} = \frac{\sin^2\frac{1}{2} u' - \sin^2\frac{1}{2} u''}{2\epsilon^2}$$

$$\sin 2X_1 = \frac{(r' - r'')(1 - e^2)a}{4er' r'' \epsilon_1^2} = \frac{\cos^2\frac{1}{2} f'' - \cos^2\frac{1}{2} f'}{2} \tag{7}$$

The equations of elliptic motion are:

$$r = a(1 - e \cos u)$$

$$r \cos f = a(\cos u - e)$$

$$r \sin f = a\sqrt{1 - e^2} \sin u$$

$$n\,dt = \frac{r}{a}\,du \tag{8}$$

Since, by Equation (1):

$$\sin\frac{1}{2} u = \epsilon[\cos X \sin k + \sin X]$$

we write Equations (8) in terms of k.

$$r = a\left(1 - e + e\epsilon^2 + e\epsilon^2 \sin^2 X\right) + 2ae\epsilon^2 \sin 2X \sin k - ae\epsilon^2 \cos^2 X \cos 2k$$

$$r \cos f = a\left(1 - e - \epsilon^2 - \epsilon^2 \sin^2 X\right) - 2a\epsilon^2 \sin 2X \sin k + a\epsilon^2 \cos^2 X \cos 2k$$

$$r \sin f = 2a\epsilon\sqrt{1 - e^2}(\cos X \sin k + \sin X)\, A$$

$$n\,dt = \frac{r}{a}\left(\frac{2\epsilon \cos X \cos k\, dk}{A}\right)$$

$$A = \left(1 - \epsilon^2 \sin^2 X - \epsilon^2 \sin 2X \sin k - \epsilon^2 \cos^2 X \sin^2 k\right)^{1/2} \tag{9}$$

On the other hand, the elliptic equations may be written as:

$$
\begin{aligned}
r &= \frac{a\left(1-e^2\right)}{1+e\cos f} \\
n\,dt &= \frac{r^2}{a^2\sqrt{1-e^2}}\,df
\end{aligned}
\tag{10}
$$

and by Equations (1):

$$\cos\frac{1}{2}f = \epsilon_1\left(\cos X_1 \sin k_1 - \sin X_1\right)$$

we write Equations (8) in terms of $k_1$.

$$
\begin{aligned}
\frac{1}{r} &= \frac{\left(1-e-e\epsilon_1^2+e\epsilon_1^2\sin^2 X_1\right) - 2e\epsilon_1^2\sin 2X_1\sin k_1 - e\epsilon_1^2\cos^2 X_1\cos 2k_1}{a\left(1-e^2\right)} \\
\cos f &= \left(\epsilon_1^2+\epsilon_1^2\sin^2 X_1 - 1\right) - 2\epsilon_1^2\sin 2X_1\sin k_1 - \epsilon_1^2\cos^2 X_1\cos 2k_1 \\
\sin f &= 2\epsilon_1\left(\cos X_1\sin k_1 - \sin X_1\right)B \\
n\,dt &= -\frac{r^2}{a^2\sqrt{1-e^2}}\cdot\frac{2\epsilon_1\cos X_1\cos k_1\,dk_1}{B} \\
B &\equiv \left(1-\epsilon_1^2\sin^2 X_1+\epsilon_1^2\sin 2X_1\sin k_1-\epsilon_1^2\cos^2 X_1\sin^2 k_1\right)^{1/2}
\end{aligned}
\tag{11}
$$

The equations of elliptic motion are now expressed in terms of the partial anomalies. When $a$ and $e$ are known, we can calculate $\epsilon$, $\epsilon_1$, $X$, and $X_1$ for any pair of values of $r'$ and $r''$.

The first of Equations (9) gives:

$$r = a\left(1-e+e\epsilon^2+e\epsilon^2\sin^2 X\right) + 2ae\epsilon^2\sin 2X\sin k - ae\epsilon^2\cos^2 X\cos 2k$$

Consequently:

$$\frac{dr}{dk} = 2ae\epsilon^2\left(\sin 2X\cos k + \cos^2 X\sin 2k\right)$$

The maxima of $r$ are at:

$$\frac{dr}{dk} = 0$$

which gives:

$$k = 90^\circ$$

and

$$k = 270^\circ$$

These values give:

$$r = r' \text{ at } k = 90^\circ$$

$$r = r'' \text{ at } k = 270^\circ$$

The minimum value of $r$ is at:

$$\sin k = -\tan X$$

which gives:

$$r = a(1 - e)$$

Similarly, using the first of Equations (11):

$$k_1 = 90^\circ \text{ at } r = r'$$

$$k_1 = 270^\circ \text{ at } r = r''$$

where $r'$ and $r''$ are now minima of $r$. The maximum of $r$ is at:

$$r = a(1 + e)$$

The ellipse has been broken into two parts. The partial anomaly $k$ represents the portion of the ellipse from perihelion to $r = r'$ on one side of the major axis and from perihelion to $r = r''$ on the other side. The partial anomaly $k_1$ represents the ellipse from aphelion to $r = r'$ on one side of the major axis and from aphelion to $r = r''$ on the other side. At $r = r''$, $k = 270^\circ$. As $r$ decreases, $k$ increases. At perihelion, $k = 0^\circ$. For $270^\circ \leq k \leq 360^\circ$, we have the true anomaly, $f$, greater than $180^\circ$. After passing perihelion, $r$ increases as does $k$ until at $r = r'$ we have $k = 90^\circ$. For $0^\circ \leq k \leq 90^\circ$, we have $f \leq 180^\circ$. If we allow $k$ to increase beyond $90^\circ$ we retrace the same portion of the ellipse in the reverse order.

At $r = r'$ we have $k_1 = 90°$. As $r$ increases, so does $k_1$. At aphelion we have $k_1 = 180°$ and finally at $r = r''$, $k_1 = 270°$. If $k_1$ increases beyond 270°, the same portion of the ellipse is retraced in the reverse order.

We see that $k$, the inferior anomaly, represents the portion of the ellipse containing perihelion and cannot represent the portion containing aphelion unless $r' = r'' = a(1+e)$, in which case we have not really divided the ellipse at all and $k$ is equivalent to the eccentric anomaly. Similarly, $k_1$ represents the portion of the ellipse containing aphelion.

By selecting one of the points of separation, $r'$ or $r''$, at perihelion or aphelion, we may divide the ellipse into three or four segments. Consider how we divide the ellipse into four parts.

Let:

$$r'' = a(1-e)$$

$$k = 2K_1 - 90$$

$$E = \sqrt{2}\,\epsilon$$

Equations (9) become:

$$r = a\left(1 - e + \frac{3}{4}\,eE^2\right) - aeE^2 \cos 2K_1 + \frac{1}{4}\,aeE^2 \cos 4K_1$$

$$r \cos f = a\left(1 - e + \frac{3}{4}\,E^2\right) + aE^2 \cos 2K_1 - \frac{1}{4}\,aE^2 \cos 4K_1$$

$$r \sin f = aE\sqrt{1-e^2}\,\left(1 - \cos 2K_1\right)\sqrt{1 - E^2 \sin^4 K_1}$$

$$n\,dt = \frac{r}{a} \cdot \frac{2E \sin 2K_1\, dK_1}{\sqrt{1 - E^2 \sin^4 K_1}} \qquad (12)$$

Equations (12) show that at perihelion $K_1 = 0°$. As $K_1$ increases, $r$ increases. At $K_1 = 90°$, we have $r = r'$. Thus, these equations represent the portion of the ellipse from perihelion to $r = r'$. The quadrant is defined by noting that Equations (12) give $f < 180°$. For values of $K_1$ greater than 90° the same segment of the ellipse is retraced.

Let:

$$r'' = a(1+e)$$

$$k_1 = 2K_2 - 90°$$

$$E_1 = \sqrt{2}\,\epsilon_1$$

Equations (11) become:

$$\frac{1}{r} = \frac{\left(1 - e + \frac{3}{4}\, eE_1^{\,2}\right) - eE_1^{\,2} \cos 2K_2 + \frac{1}{4}\, eE_1^{\,2} \cos 4K_2}{a\left(1 - e^2\right)}$$

$$\cos f = \left(\frac{3}{4}\, E_1^{\,2} - 1\right) - E_1^{\,2} \cos 2K_2 + \frac{1}{4}\, E_1^{\,2} \cos 4K_2$$

$$\sin f = E_1 \left(1 - \cos 2K_2\right) \sqrt{1 - E_1^{\,2} \sin^4 K_2}$$

$$n\,dt = -\frac{r^2}{a^2 \sqrt{1 - e^2}} \cdot \frac{2E_1 \sin 2K_2\, dK_2}{\sqrt{1 - E_1^{\,2} \sin^4 K_2}} \tag{13}$$

Equations (13) show that at $r = r'$, $K_2 = 90°$. As $K_2$ increases, so does $r$ until at aphelion, $r = a(1 + e)$, we have $K_2 = 180°$. For values of $K_2$ outside of the range $90° \leq K_2 \leq 180°$, the same segment of the ellipse is represented. We also see that $f \leq 180°$.

Let:

$$r' = a(1 + e)$$

$$k_1 = 2K_3 + 90°$$

$$E_1 = \sqrt{2}\, \epsilon_1$$

Equations (11) become:

$$\frac{1}{r} = \frac{\left(1 - e + \frac{3}{4}\, eE_1^{\,2}\right) - eE_1^{\,2} \cos 2K_3 + \frac{1}{4}\, eE_1^{\,2} \cos 4K_3}{a\left(1 - e^2\right)}$$

$$\cos f = \left(\frac{3}{4}\, E_1^{\,2} - 1\right) - E_1^{\,2} \cos 2K_3 + \frac{1}{4}\, E_1^{\,2} \cos 4K_3$$

$$\sin f = E_1 \left(\cos 2K_3 - 1\right) \sqrt{1 - E_1^{\,2} \sin^4 K_3}$$

$$n\,dt = \frac{r^2}{a^2 \sqrt{1 - e^2}} \cdot \frac{2E_1 \sin 2K_3\, dK_3}{\sqrt{1 - E_1^{\,2} \sin^4 K_3}} \tag{14}$$

Equations (14) show that at aphelion, $K_3 = 180°$. As $K_3$ increases, $r$ decreases. At $K_3 = 270°$, we have $r = r''$. We also have $f \geq 180°$.

Finally;

$$r' = a(1-e)$$

$$k = 2K_4 + 90°$$

$$E = \epsilon\sqrt{2}$$

Equations (9) become:

$$r = a\left(1 - e + \frac{3}{4}\,eE^2\right) - aeE^2 \cos 2K_4 + \frac{1}{4}\,aeE^2 \cos 4K_4$$

$$r \cos f = a\left(1 - e - \frac{3}{4}\,E^2\right) + aE^2 \cos 2K_4 - \frac{1}{4}\,aE^2 \cos 4K_4$$

$$r \sin f = aE\sqrt{1-e^2}\left(\cos 2K_4 - 1\right)\sqrt{1 - E^2 \sin^4 K_4}$$

$$n\,dt = -\frac{r}{a} \cdot \frac{2E \sin 2K_4\, dK_4}{\sqrt{1 - E^2 \sin^4 K_4}} \tag{15}$$

When $K_4 = 270°$, $r = r''$ and as $K_4$ increases $r$ decreases until at perihelion we have $K_4 = 360°$. Also, $f > 180°$.

Table I summarizes the results.

Table I

| Partial Anomaly | Range | Range of r |
|---|---|---|
| $K_1$ | 0° — 90° | perihelion to $r'$ |
| $K_2$ | 90° — 180° | $r'$ to aphelion |
| $K_3$ | 180° — 270° | aphelion to $r''$ |
| $K_4$ | 270° — 360° | $r''$ to perihelion |

The ellipse has been divided into four parts with the points of separation at $r = a(1-e)$, $r = r'$, $r = a(1+e)$, and $r = r''$.

Suppose we omit the point of separation at perihelion and take the three points of separation at $r = r'$, $r = a(1+e)$, and $r = r''$. Then Equations (9) using the inferior partial anomaly $k$

represent the portion of the ellipse from $r''$ to $r'$ containing perihelion. We have:

$$k = 270^\circ \text{ at } r = r''$$

$$k = 0^\circ \text{ at } r = a(1-e)$$

$$k = 90^\circ \text{ at } r = r'$$

The portion of the ellipse from $r'$ to $r = a(1+e)$ is represented by Equations (13) using $K_2$. We have:

$$K_2 = 90^\circ \text{ at } r = r'$$

$$K_2 = 180^\circ \text{ at } r = a(1+e)$$

The portion of the ellipse from $a(1+e)$ to $r''$ is represented by Equations (14) using $K_3$. We have:

$$K_3 = 180^\circ \text{ at } r = a(1+e)$$

$$K_3 = 270^\circ \text{ at } r = r'' .$$

Thus, the ellipse has been divided into three sections.

The partial anomalies $k$ and $k_1$ and the related anomalies $K_1$, $K_2$, $K_3$, and $K_4$ require that the points of separation be on opposite sides of the major axis or at the end points of the major axis. We cannot take two points of separation on the same side of the major axis. Since it may be necessary to select two or more points of separation on the same side of the major axis, we define the intermediate anomalies which permit the number and position of the points of separation to be completely arbitrary.

Let $r_1$ and $r_2$ be any two radius vectors of the ellipse on the same side of the major axis. For definiteness, let $r_2 \geq r_1$. Define the intermediate anomalies $L_1$ and $L_4$ by:

$$\cos K_1 = \ell \cos L_1$$

$$\cos K_4 = \ell \cos L_4 \tag{16}$$

and define $E$ by:

$$E = \left[\frac{r_2 - a(1-e)}{2ae}\right]^{1/2} \tag{17}$$

The modulus $\ell$ will be defined later. Equations (12) become:

$$r = a\left[1 - e + eE^2\left(2 - 2\ell^2 + \frac{3}{4}\ell^4\right)\right] - aeE^2\left(2\ell^2 - \ell^4\right)\cos 2L_1 + \frac{1}{4}aeE^2\ell^4\cos 4L_1$$

$$r\cos f = a\left[1 - e - E^2\left(2 - 2\ell^2 + \frac{3}{4}\ell^4\right)\right] + aE^2\left(2\ell^2 - \ell^4\right) - \frac{1}{4}aE^2\ell^4\cos 4L_1$$

$$r\sin f = B'\, aE\sqrt{1 - e^2}\left[\left(2 - \ell^2\right) - \ell^2\cos 2L_1\right]$$

$$n\,dt = \frac{r}{a}\cdot\frac{2E\ell^2\sin 2L_1\, dL_1}{B'} \tag{18}$$

where:

$$B' \equiv \left[1 - E^2\left(1 - \ell^2 + \frac{3}{8}\ell^4\right) + E^2\left(\ell^2 - \frac{1}{2}\ell^4\right)\cos 2L_1 - \frac{1}{8}E^2\ell^4\cos 4L_1\right]^{1/2}$$

Equations (15) become:

$$r = a\left[1 - e + eE^2\left(2 - 2\ell^2 + \frac{3}{4}\ell^4\right)\right] - aeE^2\left(2\ell^2 - \ell^4\right)\cos 2L_1 + \frac{1}{4}aeE^2\ell^4\cos 4L_4$$

$$r\cos f = a\left[1 - e - E^2\left(2 - 2\ell^2 + \frac{3}{4}\ell^4\right)\right] + aE^2\left(2\ell^2 - \ell^4\right)\cos 2L_4 - \frac{1}{4}aE^2\ell^4\cos 4L_4$$

$$r\sin f = aE\sqrt{1 - e^2}\left[\ell^2\cos 2L_4 - \left(2 - \ell^2\right)\right]C'$$

$$n\,dt = -\frac{r}{a}\cdot\frac{2E\ell^2\sin 2L_4\, dL_4}{C'} \tag{19}$$

where:

$$C' \equiv \left[1 - E^2\left(1 - \ell^2 + \frac{3}{8}\ell^4\right) + E^2\left(\ell^2 - \frac{1}{2}\ell^4\right)\cos 2L_4 - \frac{1}{8}E^2\ell^4\cos 4L_2\right]^{1/2}$$

The expressions for $L_1$ are used if the values of $f_1$ and $f_2$, corresponding to $r_1$ and $r_2$, are less than 180°. If $f_1$ and $f_2$ are greater than 180°, the expressions for $L_4$ are used. Equations (18) give $r = r_2$ at $L_1 = 90°$ while Equations (19) give $r = r_2$ at $L_4 = 270°$. Since $r_2$ is the maximum value of $r$ to which Equations (18) and (19) apply, other values of $L$ must pertain to values of $r$ less than $r_2$. Since the equation for $r\sin f$ in (18) and (19) shows that $\sin f$ cannot change sign, all of the values of $r$ represented must lie on the same side of the major axis whatever $L$ may be.

Obviously, we wish $r = r_1$ to be the minimum value of $r$ represented. We thus define the modulus $\ell$ of Equations (16) such that at $r = r_1$, we have $L = 0°$; that is we have $L_1 = 0°$ or $L_4 = 360°$. If $L = 0°$ the first of Equations (18) or (19) gives:

$$r = r_1 = a\left[1 - e + eE^2\left(2 - 4\ell^2 + 2\ell^4\right)\right]$$

This gives:

$$\ell^2 = 1 - \left[\frac{r_1 - a(1-e)}{r_2 - a(1-e)}\right]^{1/2} \qquad (20)$$

Define $\omega$ by:

$$\tan(45° - \omega) = \left[\frac{r_1 - a(1-e)}{r_2 - a(1-e)}\right]^{1/2} \qquad (21)$$

Then:

$$\ell^2 = \frac{\sin\omega}{\cos(45° - \omega)}\sqrt{2} \qquad (22)$$

This definition of $\ell$ gives $r = r_1$ for $L_1 = 0°$ or $L_4 = 360°$. The value $r = r_2$ occurs for $L_1 = 90°$ or $L_4 = 270°$. Values of $L_1$ outside of the range $0° \le L_1 \le 90°$ or values of $L_4$ outside of the range $270° \le L_4 \le 360°$ simply retrace the same segment of the ellipse between $r = r_1$ and $r = r_2$. Consequently, $L_1$ represents the segment of the ellipse from $r = r_1$ to $r = r_2$ if $f < 180°$ and $L_4$ represents the segment of the ellipse from $r = r_1$ to $r = r_2$ if $f > 180°$. Both $r_1$ and $r_2$ must be on the same side of the major axis if we are to apply the intermediate anomalies. If they are not, of course, we would apply the partial anomalies $k$ and $k_1$.

We may also define $L_2$ and $L_3$.

$$\cos K_2 = \ell_1 \cos L_2$$

$$\cos K_3 = \ell_1 \cos L_3 \qquad (23)$$

Define:

$$E_1 = \left[\frac{a(1+e) - r_1}{2er_1}(1-e)\right]^{1/2} \qquad (24)$$

We could then write Equations (13) and (14) in terms of $L_2$ and $L_3$, respectively. We then define $\ell_1$ as:

$$\ell_1^2 \equiv \frac{\sin \omega_1}{\cos\left(45^\circ - \omega_1\right)} \sqrt{2} \tag{25}$$

where:

$$\tan\left(45^\circ - \omega_1\right) \equiv \left[\frac{a(1+e) - r_2}{a(1+e) - r_1} \cdot \frac{r_1}{r_2}\right]^{1/2} \tag{26}$$

We would then find that $r = r_1$ for $L_2 = 90^\circ$ or $L_3 = 270^\circ$. The value $r = r_2$ occurs for $L_2 = 180^\circ$ or $L_3 = 180^\circ$. Values of $L_2$ outside of the range $90^\circ \le L_2 \le 180^\circ$ or values of $L_3$ outside of the range $180^\circ \le L_3 \le 270^\circ$ will retrace the same segment of the ellipse between $r = r_1$ and $r = r_2$. Consequently, $L_2$ represents the segment of the ellipse from $r = r_1$ to $r = r_2$ if $f < 180^\circ$ and $L_3$ represents this segment if $f > 180^\circ$.

As an example, consider six points of separation at $r = a(1-e)$, $r = r_1$, $r = r_2$, $r = a(1+e)$, $r = r_1'$, and $r = r_2'$. Figure I shows the situation. The application of the various partial anomalies is given in Table II.

We could divide the ellipse into more than six parts by defining more intermediate anomalies and moduli in the same manner.

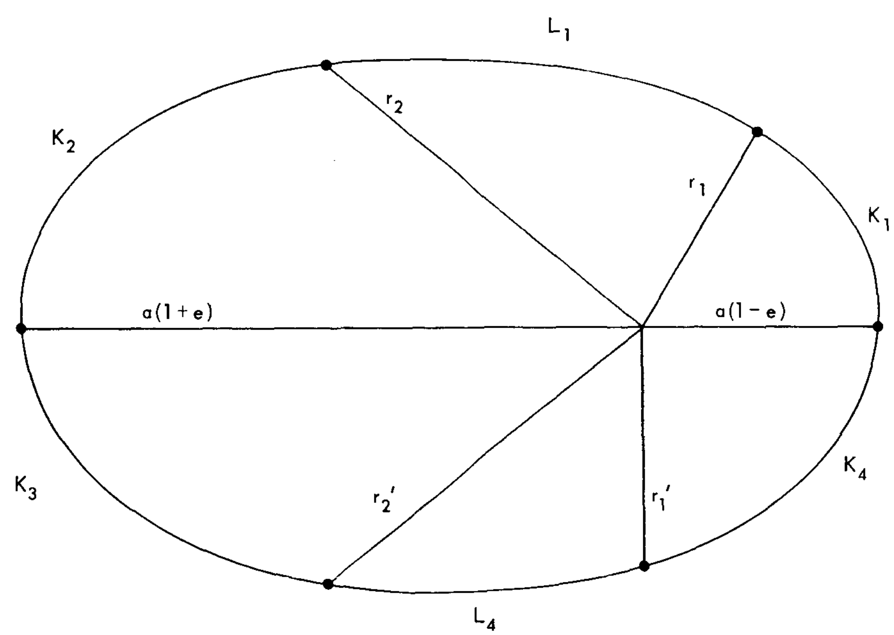


Figure I—Division of the ellipse into six sections.

Table II

| Partial Anomaly | Range | Range of r |
|---|---|---|
| $K_1$ | 0° — 90° | perihelion to $r_1$ |
| $L_1$ | 0° — 90° | $r_1$ to $r_2$ |
| $K_2$ | 90° — 180° | $r_2$ to aphelion |
| $K_3$ | 180° — 270° | aphelion to $r_2'$ |
| $L_4$ | 270° — 360° | $r_2'$ to $r_1'$ |
| $K_4$ | 270° — 360° | $r_1'$ to perihelion |

In applying the partial anomalies to the calculation of perturbations, the various radicals appearing must be developed in a Fourier series of multiples of the partial anomaly. This develepement will be made so that very convergent series for the perturbations are obtained. The manner in which we select the number and position of the points of separation will depend on the specific problem. They will always be selected to make the various moduli, $\epsilon$, $\epsilon_1$, E, $E_1$, $\ell$, or $\ell_1$, as small as possible since the smaller the modulus the more convergent the series obtained will be.

Consequently, by dividing the ellipse properly, the series representing the perturbations may be made to converge rapidly. A different series is used in each segment of the ellipse. If the ellipse is divided into n parts, we will have n sets of series for the perturbations. But the convergence will be much more rapid than that of the series which would represent the whole ellipse using the mean anomaly or one of the other conventional anomalies.

## II. DEVELOPMENT OF THE RADICALS

In Equations (9), (11), (12), and (18), radicals of the following form appear:

$$\left(1 - \epsilon^2 \sin^2 k\right)^{\pm 1/2} \tag{27a}$$

$$\left(1 - E^2 \sin^4 K\right)^{\pm 1/2} \tag{27b}$$

$$\left(1 + \sin^2 K\right)^{\pm 1/2} \tag{27c}$$

$$\left(A + B \sin k + C \sin^2 k\right)^{\pm 1/2} \tag{27d}$$

$$\left(A' + B' \cos 2L + C' \cos 4L\right)^{\pm 1/2} \tag{27e}$$

We shall be able to write all of these in a form involving only factors of the form of Equation (27a). Consequently, consider the expansion of Equation (27a). Let:

$$\left(1 - \epsilon^2 \sin^2 k\right)^{-1/2} = \alpha_0 - 2\alpha_2 \cos 2k + 2\alpha_4 \cos 4k - \cdots \tag{28}$$

where:

$$\alpha_{2i} = \pm \frac{2}{\pi} \int_0^{\pi/2} \frac{\cos 2ik \, dk}{\left(1 - \epsilon^2 \sin^2 k\right)^{1/2}} \tag{29}$$

The plus sign is for $i$ even and the minus sign for $i$ odd.

The following recursion formula may be obtained by integrating by parts or by developing the integrand as a power series in $k$ and comparing terms for three successive values of $i$.

$$0 = (2i - 1)\, \epsilon^2 \alpha_{2i-2} - 4i\left(2 - \epsilon^2\right) \alpha_{2i} + (2i + 1)\, \epsilon^2 \alpha_{2i+2} \tag{30}$$

If:

$$\left(1 - \epsilon^2 \sin^2 k\right)^{+1/2} = \beta_0 - 2\beta_2 \cos 2k + 2\beta_4 \cos 4k - \cdots \tag{31}$$

then:

$$\beta_{2i} = -\frac{\epsilon^2}{8i} \left(\alpha_{2i-2} - \alpha_{2i+2}\right) \tag{32}$$

except:

$$\beta_0 = \left(1 - \frac{1}{2}\epsilon^2\right)\alpha_0 - \frac{1}{2}\epsilon^2\alpha_2 \tag{32}$$

Equation (32) may be obtained by recognizing that:

$$\left(1 - \epsilon^2 \sin^2 k\right)^{-1/2}\left(1 - \epsilon^2 \sin^2 k\right)^{+1/2} = 1$$

The calculation of $\alpha_4$, $\alpha_6$, etc. from $\alpha_0$ and $\alpha_2$ by means of Equation (30) is inaccurate as errors in the last significant figures of $\alpha_0$ and $\alpha_2$ cause larger and larger errors in $\alpha_{2i}$ as $i$ increases. To circumvent this problem, the following scheme is provided.

Define:

$$p_{2i} \equiv \frac{\alpha_{2i}}{\alpha_{2i-2}}$$

$$\gamma_{2i} \equiv \frac{4i\left(2 - \epsilon^2\right)}{(2i-1)\epsilon^2}\, p_{2i}$$

$$\lambda_{2i} \equiv \frac{(2i-1)^2}{16i(i-1)}\left(\frac{\epsilon^2}{2-\epsilon^2}\right)^2 \tag{33}$$

Equation (30) becomes:

$$0 = 1 - \gamma_{2i} + \gamma_{2i}\,\gamma_{2i+2}\,\lambda_{2i+2} \tag{34}$$

Equation (33) show that as $i$ increases, the quantity $\gamma_{2i+2} - \gamma_{2i}$ decreases. At $i = \infty$, $\gamma_{2i} = \gamma_{2i+2} = \gamma_\infty$ also:

$$\lambda_\infty = \frac{1}{4}\left(\frac{\epsilon^2}{2-\epsilon^2}\right)^2$$

Equation (34) becomes:

$$\frac{1}{4}\left(\frac{\epsilon^2}{2-\epsilon^2}\right)^2 \gamma_\infty^2 - \gamma_\infty + 1 = 0$$

or:

$$\gamma_\infty = \frac{2\left(2-\epsilon^2\right)}{\epsilon^4}\left[\left(2-\epsilon^2\right) \pm 2\sqrt{1-\epsilon^2}\right] \tag{35}$$

The (-) sign in front of the radical must be used as the (+) sign leads to divergent results.

Define:

$$\epsilon \equiv \sin\psi \tag{36}$$

Equations (2) lead to:

$$\cos\psi = \tan\left(45 - \frac{1}{2}X\right) \tag{37}$$

Equation (35) becomes:

$$\gamma_\infty = \sec^2 \frac{1}{2} X \tag{38}$$

Equation (38) is exact for $i = \infty$. The definition of $\psi$ allows us to write Equation (33) as:

$$p_{2i} = \frac{2i-1}{4i} \sin X \gamma_{2i}$$

$$\lambda_{2i} = \frac{2i-1^{\;2}}{16i(i-1)} \sin^2 X \tag{39}$$

We proceed as follows. Select some large value of $i = i_f$. Compute $\lambda_{2i}$ for $i = 2, 3, 4, \cdots i_f$ by Equations (39). Then, assume $\gamma_{2i_f} = \gamma_\infty$. We calculate $\gamma_\infty$ by Equation (38). Then, Equation (34) is written as:

$$\gamma_{2i-2} = \frac{1}{1-\lambda_{2i}\gamma_{2i}}$$

For $i = i_f$, use $\gamma_{2i} = \gamma_\infty$ and calculate $\gamma_{2i-2}$. Then calculate $\gamma_{2i-4}, \gamma_{2i-6}, \cdots, \gamma_2$. The first of Equations (39) is used to calculate $p_2, p_4, \cdots p_{2i}$. By Equation (33):

$$\alpha_{2i} = \alpha_0 p_2 p_4 \cdots p_{2i} \tag{40}$$

In this manner we compute all of the $\alpha$'s from $\alpha_0$. The equation $\gamma_{2i} = \gamma_\infty$ is only approximate for finite $i$. The accuracy of a given calculation can be estimated by recalculating the $\gamma$'s for a larger value of $i_f$ and noting the effect on the values of the $\alpha$'s.

We may improve the approximation to $\gamma_{2i}$ in the following manner. Recall that:

$$0 = 1 - \gamma_{2i} + \gamma_{2i}\,\gamma_{2i+2}\lambda_{2i+2}$$

and:

$$\lambda_{2i} = \frac{(2i-1)^2}{16i(i-1)} \sin^2 X$$

or:

$$\lambda_{2i+2} = \frac{(2i+1)^2}{16i(i+1)} \sin^2 X$$

Using this expression for $\lambda_{2i+2}$ and $\sec^2 1/2X$ for $\gamma_{2i+2}$, we have:

$$\gamma_{2i} = \left[1 - \frac{(2i+1)^2}{4i(i+1)} \sin^2 \frac{1}{2} X\right]^{-1}$$

Developing this expression and dropping powers of $\sin 1/2X$ higher than the second, we have:

$$\gamma_{2i} = \sec^2 \frac{1}{2} X \left[1 + \frac{1}{4i(i+1)} \sin^2 \frac{1}{2} X\right] \tag{41}$$

Since $X \leq 45°$, the second term is always less than 1/1000 of the first term for $i = 10$. Consequently, this approximation is sufficient for all cases of interest for moderate values of $i$.

It remains to calculate $\alpha_0$. Equation (29) gives:

$$\alpha_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dk}{\left(1 - \epsilon^2 \sin^2 k\right)^{1/2}}$$

or:

$$\alpha_0 = \frac{2}{\pi} f\left(\epsilon, \frac{\pi}{2}\right) \tag{42}$$

$f(\epsilon, \pi/2)$ is the complete elliptic integral of the first kind. It has been extensively tabulated. It may be calculated from the following series if $\epsilon^2$ is sufficiently small.

$$\frac{2}{\pi} f\left(\epsilon, \frac{\pi}{2}\right) = \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \epsilon^{2n} \tag{43}$$

This series is easily obtained by expanding $\left(1 - \epsilon^2 \sin^2 k\right)^{-1/2}$ by the binominal theorem and integrating the series term by term. The series given by Equation (43) will converge fairly rapidly if $\epsilon^2 \leq 1/2$.

Consider now the form of the radical given by Equation (27b). We may write:

$$\begin{aligned}\left(1 - E^2 \sin^4 K\right)^{-1/2} &= \left(1 + E^2 \sin^2 K\right)^{-1/2} \left(1 - E^2 \sin^2 K\right)^{1/2} \\ &= \lambda_0 + 2\lambda_2 \cos 2K + 2\lambda_4 \cos 4K + \cdots \end{aligned} \tag{44}$$

and:

$$\left(1 + E \sin^2 K\right)^{-1/2} = \theta_0 + 2\theta_2 \cos 2K + 2\theta_4 \cos 4K + \cdots \tag{45a}$$

$$\left(1 - E \sin^2 K\right)^{-1/2} = \eta_0 - 2\eta_2 \cos 2K + 2\eta_4 \cos 4K - \cdots \tag{45b}$$

By multiplication of series:

$$\begin{aligned}\lambda_0 &= \eta_0 \theta_0 - 2\eta_2 \theta_2 + 2\eta_4 \theta_4 - \cdots \\ -\lambda_2 &= \eta_2 \theta_0 - \left(\eta_0 + \eta_4\right) \theta_2 + \left(\eta_2 + \eta_6\right) \theta_4 - \cdots \\ \lambda_4 &= \eta_4 \theta_0 - \left(\eta_2 + \eta_6\right) \theta_2 + \left(\eta_0 + \eta_8\right) \theta_4 - \cdots \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned} \tag{46}$$

Equation (45b) is in the same form as Equation (28) and may be handled in exactly the same manner if we put:

$$\begin{aligned}\sqrt{E} &\equiv \sin \psi \\ \eta_{2i} &= \eta_0 \, p_2 \, p_4 \cdots p_{2i}\end{aligned} \tag{47}$$

We also have:

$$\eta_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dK}{\left(1 - E \sin^2 K\right)^{1/2}}$$

so that:

$$\eta_0 = \frac{2}{\pi} f\left(\sqrt{E}, \frac{\pi}{2}\right) \tag{48}$$

Thus, we may calculate the $\eta$'s.

Now consider Equation (45a). Define:

$$K \equiv 90^\circ - P$$

Then:

$$1 + E \sin^2 K = (1+E)\left(1 - \frac{E}{1+E}\sin^2 P\right) \tag{49}$$

This is now of the same form as Equation (28) and is dealt with accordingly. We may now calculate the $\theta$'s and by Equations (46), the $\lambda$'s.

Equation (27c) may be changed to the form:

$$\left(1 + \sin^2 K\right)^{-1/2} = \omega_0 + \omega_2 \cos 2K + \omega_4 \cos 4K + \cdots \tag{50}$$

Define:

$$K = 90^\circ - P$$

Then:

$$\left(1 + \sin^2 K\right)^{-1/2} = \left(1 + \cos^2 P\right)^{-1/2} = 2\left(1 - \frac{1}{2}\sin^2 P\right)^{-1/2} \tag{51}$$

This is again of the same form as Equation (28) and we may calculate the $\omega$'s.

Consider the form of Equation (27d). Write:

$$\left(A + B\sin k - C\sin^2 k\right)^{-1/2} = \lambda_0 + 2\lambda_1 \sin k + 2\lambda_2 \cos 2k + \cdots \tag{52a}$$

$$\left(A + B\sin k - C\sin^2 k\right)^{+1/2} = \mu_0 + 2\mu_1 \sin k + 2\mu_2 \cos 2k + \cdots \tag{52b}$$

we will have:

$$\lambda_i = \frac{1}{\pi}\int_0^\pi \frac{\cos ik \, dk}{\left(A + B\sin k - C\sin^2 k\right)^{1/2}}$$

$$\mu_i = \frac{1}{\pi}\int_0^\pi \left(A + B\sin k - C\sin^2 k\right)^{1/2} \cos ik \, dk \tag{53}$$

for i even. For i odd:

$$\lambda_i = \frac{1}{\pi}\int_0^{\pi} \frac{\sin ik\, dk}{\left(A + B\sin k - C\sin^2 k\right)^{1/2}}$$

$$\mu_i = \frac{1}{\pi}\int_0^{\pi} \left(A + B\sin k - C\sin^2 k\right)^{1/2} \sin ik\, dk \tag{54}$$

Since:

$$\left(A + B\sin k - C\sin^2 k\right)^{-1/2}\left(A + B\sin k - C\sin^2 k\right)^{+1/2} \equiv 1$$

we have:

$$\mu_i = \frac{C}{4i}\left(\lambda_{i-2} - \lambda_{i+2}\right) - \frac{B}{4i}\left(\lambda_{i-1} + \lambda_{i+1}\right)$$

and:

$$\mu_i = \frac{C}{4i}\left(\lambda_{i-2} - \lambda_{i+2}\right) + \frac{B}{4i}\left(\lambda_{i-1} + \lambda_{i+1}\right) \tag{55a}$$

for i even and odd, respectively. The exception is:

$$\mu_0 = \left(A - \frac{1}{2}C\right)\lambda_0 + B\lambda_1 + C\lambda_2 \tag{55b}$$

The coefficients A, B, and C will be given by:

$$A \equiv 1 - \epsilon^2 \sin^2 X$$

$$B \equiv \epsilon^2 \sin 2X$$

$$C \equiv \epsilon^2 \cos^2 X \tag{56}$$

Then:

$$\left(A + B\sin k - C\sin^2 k\right)^{-1/2} = \left(1 - \epsilon^2\sin^2 X + \epsilon^2 \sin 2X \sin k - \epsilon^2\cos^2 X\sin^2 k\right)^{-1/2}$$

$$= \left(1 - \epsilon\sin X + \epsilon\cos X\sin k\right)^{-1/2}\left(1 + \epsilon\sin X - \epsilon\cos X\sin k\right)^{-1/2} \tag{57}$$

Define:

$$\left(1 - \epsilon \sin X + \epsilon \cos X \sin k\right)^{-1/2} = \theta_0 - 2\theta_1 \sin k - 2\theta_2 \cos 2k + 2\theta_3 \sin 3k + 2\theta_4 \cos 4k - \cdots \tag{58a}$$

and:

$$\left(1 + \epsilon \sin X - \epsilon \cos X \sin k\right)^{-1/2} = \eta_0 + 2\eta_1 \sin k - 2\eta_2 \cos 2k - 2\eta_3 \sin 3k + 2\eta_4 \cos 4k + \cdots \tag{58b}$$

Equation (52a) and Equations (58) give:

$$\begin{aligned}
\lambda_0 &= \eta_0 \theta_0 - 2\theta_1 \eta_1 + 2\theta_2 \eta_2 - 2\theta_3 \eta_3 + 2\theta_4 \eta_4 - \cdots \\
\lambda_1 &= \eta_1 \theta_0 - \theta_1 \left(\eta_0 + \eta_2\right) + \theta_2 \left(\eta_1 + \eta_3\right) - \theta_3 \left(\eta_2 + \eta_4\right) + \theta_4 \left(\eta_3 + \eta_5\right) - \cdots \\
-\lambda_2 &= \eta_2 \theta_0 - \theta_1 \left(\eta_1 + \eta_3\right) + \theta_2 \left(\eta_0 + \eta_4\right) - \theta_3 \left(\eta_1 + \eta_5\right) + \theta_4 \left(\eta_2 + \eta_6\right) - \cdots \\
-\lambda_3 &= \eta_3 \theta_0 - \theta_1 \left(\eta_2 + \eta_4\right) + \theta_2 \left(\eta_1 + \eta_5\right) - \theta_3 \left(\eta_0 + \eta_6\right) + \theta_4 \left(\eta_1 + \eta_7\right) - \cdots \\
\lambda_4 &= \eta_4 \theta_0 - \theta_1 \left(\eta_3 + \eta_5\right) + \theta_2 \left(\eta_2 + \eta_6\right) - \theta_3 \left(\eta_1 + \eta_7\right) + \theta_4 \left(\eta_0 + \eta_8\right) - \cdots \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned} \tag{59}$$

To determine $\theta_i$ let:

$$k = 2\ell + 90° \tag{60a}$$

and to determine $\eta_i$ let:

$$k = 2\ell' - 90° \tag{60b}$$

Then:

$$1 - \epsilon \sin X + \epsilon \cos X \sin k = \left(1 - \epsilon \sin X + \epsilon \cos X\right) \left(1 - f^2 \sin^2 \ell\right) \tag{61a}$$

and

$$1 + \epsilon \sin X - \epsilon \cos X \sin k = \left(1 + \epsilon \sin X - \epsilon \cos X\right) \left(1 - f'^2 \sin^2 \ell'\right) \tag{61b}$$

where:

$$f \equiv \left[\frac{2\epsilon \cos X}{1 - \epsilon \sin X + \epsilon \cos X}\right]^{1/2} \tag{62a}$$

$$f' \equiv \left[\frac{2\epsilon \cos X}{1 + \epsilon \sin X - \epsilon \cos X}\right]^{1/2} \tag{62b}$$

Thus, we have again obtained the form of Equation (28) and may compute $\theta_i$ and $\eta_i$. The $\lambda$'s follow from Equation (59).

Equations (58) give:

$$\theta_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dk}{\left[(1 - \epsilon \sin X) + \epsilon \cos X \sin k\right]^{1/2}} \tag{63a}$$

$$\eta_0 = \frac{2}{\pi} \int \frac{dk}{\left[(1 + \epsilon \sin X) - \epsilon \cos X \sin k\right]^{1/2}} \tag{63b}$$

Define:

$$a \equiv 1 - \epsilon \sin X$$

$$b \equiv \epsilon \cos X$$

$$a' \equiv 1 + \epsilon \sin X \tag{64}$$

Since $-45° \leq X \leq +45°$, $a$, $b$, and $b'$ are always greater than zero. Then:

$$\theta_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dk}{(a + b \sin k)^{1/2}} \tag{65a}$$

$$\eta_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dk}{(a' - b \sin k)^{1/2}} \tag{65b}$$

Equation 288.00 of Byrd and Friedman (Reference 2) gives for $a > b > 0$:

$$\frac{\pi}{2} \theta_0 = g f(\xi, q) \tag{66}$$

where:

$$g = \frac{2}{\sqrt{a + b}}$$

$$q^2 = \frac{2b}{a + b}$$

$$\xi = \sin^{-1}\sqrt{\frac{1}{2}} \tag{67}$$

and $f(\xi, q)$ is the incomplete elliptic integral of the first kind. If $b > a > 0$, Equation 288.50 of Reference 2 gives:

$$\frac{\pi}{2}\theta_0 = g f(\psi, q) \tag{68}$$

where:

$$g = \sqrt{\frac{2}{b}}$$

$$q^2 = \frac{a + b}{2b}$$

$$\psi = \sin^{-1}\sqrt{\frac{b}{a + b}} \tag{69}$$

Define:

$$k = 90^\circ - T$$

then:

$$\eta_0 = \frac{2}{\pi}\int_0^{\pi/2} \frac{dT}{(a' - b\cos T)^{1/2}} \tag{70}$$

By the last of Equations (64), $a' > b$. For $a' > b > 0$, Equation 291.00 of Reference 2 gives:

$$\frac{\pi}{2}\eta_0 = g f(A, q) \tag{71}$$

where:

$$g \equiv \frac{2}{\sqrt{a' + b}}$$

$$q^2 \equiv \frac{2b}{a' + b}$$

$$A \equiv \sin^{-1}\sqrt{\frac{b}{q^2 (a' - b)}} \tag{72}$$

The incomplete elliptic integral of the first kind is extensively tabulated.

Finally, the form in Equation (27e) was:

$$A' + B' \cos 2L + C' \cos 4L$$

In our particular case:

$$A' \equiv 1 - E^2\left(1 - \ell^2 + \frac{3}{8}\ell^4\right)$$

$$B' \equiv E^2\left(\ell^2 - \frac{1}{2}\ell^4\right)$$

$$C' \equiv -\frac{1}{8}E^2\ell^4 \tag{73}$$

we may write:

$$\left(A' + B' \cos 2L + C' \cos 4L\right)^{-1/2} = \left(1 + E - \ell^2 E \cos^2 L\right)^{-1/2} \left(1 - E + \ell^2 E \cos^2 L\right)^{-1/2}$$

$$= \lambda_0 + 2\lambda_2 \cos 2L + 2\lambda_4 \cos 4L + \cdots \tag{74}$$

write:

$$\left(1 + E - \ell^2 E \cos^2 L\right)^{-1/2} = \eta_0 - 2\eta_2 \cos 2L + 2\eta_4 \cos 4L - \cdots$$

$$\left(1 - E + \ell^2 E \cos^2 L\right)^{-1/2} = \theta_0 + 2\theta_2 \cos 2L + 2\theta_4 \cos 4L + \cdots \tag{75}$$

The $\eta$'s and $\theta$'s are related to the $\lambda$'s as in Equation (46). The second factor of Equation (74) may be written as:

$$\left(1 - E + \ell^2 E \cos^2 L\right) = \left(1 - E + \ell^2 E\right)\left(1 - \frac{\ell^2 E}{1 - E + \ell^2 E}\sin^2 L\right) \tag{76}$$

which is in the form of Equation (28). The first factor of Equation (74) is:

$$\left(1 + E - \ell^2 E \cos^2 L\right) = (1 + E)\left(1 - \frac{\ell^2 E}{1 + E} \cos^2 L\right)$$

Define:

$$L \equiv 90^\circ - T \tag{77}$$

Then:

$$\left(1 + E - \ell^2 E \cos^2 L\right) = (1 + E)\left(1 - \frac{\ell^2 E}{1 + E} \sin^2 T\right)$$

and this is in the proper form.

From Equations (75):

$$\eta_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dL}{\left(1 + E - \ell^2 E \cos^2 L\right)^{1/2}} \tag{78}$$

$$\theta_0 = \frac{2}{\pi} \int_0^{\pi/2} \frac{dL}{\left(1 - E + \ell^2 E \cos^2 L\right)^{1/2}} \tag{79}$$

Equation (78) is transformed to the form of Equation (28) by the substitution, $L = 90 - T$. Equation (79) is transformed by simply writing $\cos^2 L = 1 - \sin^2 L$.

We have seen how the partial anomalies are defined and how the various radicals can be expanded in a form convenient for integration. The partial anomalies will permit us to obtain highly convergent series by means of which the perturbations of a body may be expressed. The requirement that the series be highly convergent will determine how we must select the points of separation in dividing the ellipse.

The series for computing perturbations, when expressed in the conventional anomalies, may become very slowly convergent or divergent if the radius vector of the perturbed body is at one time much larger and at another time much smaller than the radius vector of the perturbing body. Consequently, the points of separation must be selected such that in each segment of the perturbed ellipse, the radius vector of the perturbed body is always larger or always smaller than that of the perturbing body. If the two bodies approach each other closely, the small segment of the perturbed orbit near this minimum of distance should be represented by the intermediate anomaly. If the segment represented by the intermediate anomlay is small, the value of the modulus, $\ell$, will be very small and this will lead to highly convergent series.

## III. DETERMINATION OF CONSTANTS OF INTEGRATION

We now consider the types of integrations we will be performing and the determination of the constants of integration. As we have seen in Section II, the radicals involved in the various partial anomalies may be expressed in the general form:

$$\beta_0 + \beta_1 \sin k + \beta_2 \cos 2k + \beta_3 \sin 3k + \beta_4 \cos 4k + \cdots \tag{80}$$

In certain cases the coefficients of the sine terms are zero. The expressions for the polar coordinates of a body in an elliptic orbit have been given in Section I in terms of the partial anomalies and these coordinates may also be expressed in the form of Equation (80). Since the perturbation function and its derivatives involve these coordinates, they may also be expressed in this form.

Assume that the perturbed body is a comet and the perturbing body a planet. Let $g'$ be the mean anomaly of the planet. If $\gamma$ is any elliptic element, we may write:

$$d\gamma = dt \sum_{i=0}^{\infty} \left(\alpha_0 + \alpha_1 \sin k + \alpha_2 \cos 2k + \cdots\right) \begin{matrix}\sin\\ \cos\end{matrix}\, ig' \tag{81}$$

The symbol $\begin{matrix}\sin\\ \cos\end{matrix}\, ig'$ implies that a series in $\sin ig'$ applies when $\gamma$ is a particular elliptic element representable by a sine series, $\Omega$ or $\omega$, for example, and a series in $\cos ig'$ applies when $\gamma$ is a particular elliptic element representable by a cosine series, $a$, $e$, $i$, for example.

The results of Sections I and II show that $n\,dt$ may be written as:

$$n\,dt = \left(\lambda_1 \cos k + \lambda_2 \sin 2k + \lambda_3 \cos 3k + \cdots\right) dk \tag{82}$$

where $n$ is the mean motion of the comet defined by Kepler's third law. Eliminating $dt$ between Equations (81) and (82):

$$d\gamma = dk \sum \left(\kappa_1 \cos k + \kappa_2 \sin 2k + \cdots\right) \begin{matrix}\cos\\ \sin\end{matrix}\, ig' \tag{83}$$

If $n'$ is the mean motion of the planet and $c'$ is its mean anomaly at $t = 0$, we have:

$$g' = n' t + c' \tag{84}$$

Define:

$$\nu \equiv \frac{n'}{n} \tag{85}$$

Integrating Equation (82) and using Equation (85), we have:

$$n' t = \nu\lambda_0 + \nu\lambda_1 \sin k - \frac{1}{2} \nu\lambda_2 \cos 2k + \frac{1}{3} \nu\lambda_3 \sin 3k - \cdots \tag{86}$$

where $\lambda_0$ is the constant of integration. We now eliminate $g'$ from Equation (83) by means of Equation (86) and obtain:

$$d\gamma = dk \sum_{\ell=0}^{\infty} \left(\kappa_1 \cos k + \kappa_2 \sin 2k + \kappa_3 \cos 3k + \cdots\right) \begin{matrix}\sin\\ \cos\end{matrix} \left(ic' + i\nu\lambda_0 + i\nu\lambda_1 \sin k - \frac{1}{2} i\nu\lambda_2 \cos 2k + \cdots\right) \tag{87}$$

By known trigonometric identities:

$$\sin\left(ic' + i\nu\lambda_0 + \cdots\right) = \sin ic' \left(\theta_0 + \theta_1 \sin k + \theta_2 \cos 2k + \cdots\right) + \cos ic' \left(\theta_0' + \theta_1' \sin k + \theta_2' \cos 2k + \cdots\right)$$

$$\cos\left(ic' + i\nu\lambda_0 + \cdots\right) = \cos ic' \left(\theta_0 + \theta_1 \sin k + \theta_2 \cos 2k + \cdots\right) - \sin ic' \left(\theta_0' + \theta_1' \sin k + \theta_2' \cos 2k + \cdots\right)$$

Equation (87) becomes:

$$d\gamma = dk \sum \left(\omega_1 \cos k + \omega_2 \sin 2k + \omega_3 \cos 3k + \cdots\right) \begin{matrix}\sin\\ \cos\end{matrix}\, ic' \tag{88}$$

Since $c'$ is independent of the time and hence of the partial anomaly, we have:

$$\gamma = \text{const} + \sum \left(\omega_1 \sin k - \frac{1}{2} \omega_2 \cos 2k + \frac{1}{3} \omega_3 \sin 3k - \cdots\right) \begin{matrix}\sin\\ \cos\end{matrix}\, ic' \tag{89}$$

Let us select a particular perihelion passage of the comet as the epoch of time. Then, $g' = c'$ at $t = 0$, and $c_0'$ is the mean anomaly of the planet at the instant of this particular perihelion passage of the comet. Let $T$ be the time of the closest perihelion passage of the comet, past or future, from the time $t$. Then, $c'$ will be the mean anomaly of the planet at time $T$. Thus, $c'$ is fixed during each revolution of the comet but changes from revolution to revolution.

In order to proceed, we must select the points of separation. To illustrate a fairly complex example, we select six points of separation.

These points are:

$$r = a(1 - e)$$

$$r = r'$$

$$r = r_1'$$

$$r = a(1+e)$$

$$r = r''$$

$$r = r_1''$$

This is illustrated in Figure I. We have selected $f'$ and $f_1'$ less than 180° and $f''$ and $f_1''$ greater than 180°. The following partial anomalies are required.

| Segment of Ellipse | Partial Anomaly |
|---|---|
| Perihelion to $r'$ | $K_1$ |
| $r'$ to $r_1'$ | $L_1$ |
| $r_1'$ to Aphelion | $K_2$ |
| Aphelion to $r''$ | $K_3$ |
| $r''$ to $r_1''$ | $L_4$ |
| $r_1''$ to Perihelion | $K_4$ |

We write Equation (89) for each segment.

$$\gamma = \kappa_1^{(0)} + \sum\left(\kappa_1^{(2)} \cos 2K_1 + \kappa_1^{(4)} \cos 4K_1 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' \tag{90a}$$

$$\gamma = \lambda_1^{(0)} + \sum\left(\lambda_1^{(2)} \cos 2L_1 + \lambda_1^{(4)} \cos 4L_1 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' \tag{90b}$$

$$\gamma = \kappa_2^{(0)} + \sum\left(\kappa_2^{(2)} \cos 2K_2 + \kappa_2^{(4)} \cos 4K_2 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' \tag{90c}$$

$$\gamma = \kappa_3^{(0)} + \sum\left(\kappa_3^{(2)} \cos 2K_3 + \kappa_3^{(4)} \cos 4K_3 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' \tag{90d}$$

$$\gamma = \lambda_4^{(0)} + \sum\left(\lambda_4^{(2)} \cos 2L_4 + \lambda_4^{(4)} \cos 4L_4 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' \tag{90e}$$

$$\gamma = \kappa_4^{(0)} + \sum\left(\kappa_4^{(2)} \cos 2K_4 + \kappa_4^{(4)} \cos 4K_4 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' \tag{90f}$$

$\kappa_1^{(0)}$, $\lambda_1^{(0)}$, $\kappa_2^{(0)}$, $\kappa_3^{(0)}$, $\lambda_4^{(0)}$, and $\kappa_4^{(0)}$ are constants of integration. We can see from the results of Section I that for this division of the ellipse no sine terms are present.

Define:

$$M_1 = R_1 + S_1 = \kappa_1^{(2)} + \kappa_1^{(4)} + \kappa_1^{(6)} + \cdots \qquad (91a)$$

$$M_2 = R_2 + S_2 = \kappa_2^{(2)} + \kappa_2^{(4)} + \kappa_2^{(6)} + \cdots \qquad (91b)$$

$$M_3 = R_3 + S_3 = \kappa_3^{(2)} + \kappa_3^{(4)} + \kappa_3^{(6)} + \cdots \qquad (91c)$$

$$M_4 = R_4 + S_4 = \kappa_4^{(2)} + \kappa_4^{(4)} + \kappa_4^{(6)} + \cdots \qquad (91d)$$

$$P_1 = T_1 + U_1 = \lambda_1^{(2)} + \lambda_1^{(4)} + \lambda_1^{(6)} + \cdots \qquad (91e)$$

$$P_4 = T_4 + U_4 = \lambda_4^{(2)} + \lambda_4^{(4)} + \lambda_4^{(6)} + \cdots \qquad (91f)$$

$$N_1 = R_1 - S_1 = \kappa_1^{(2)} - \kappa_1^{(4)} + \kappa_1^{(6)} - \cdots \qquad (91g)$$

$$N_2 = R_2 - S_2 = \kappa_2^{(2)} - \kappa_2^{(4)} + \kappa_2^{(6)} - \cdots \qquad (91h)$$

$$N_3 = R_3 - S_3 = \kappa_3^{(2)} - \kappa_3^{(4)} + \kappa_3^{(6)} - \cdots \qquad (91i)$$

$$N_4 = R_4 - S_4 = \kappa_4^{(2)} - \kappa_4^{(4)} + \kappa_4^{(6)} - \cdots \qquad (91j)$$

$$Q_1 = T_1 - U_1 = \lambda_1^{(2)} - \lambda_1^{(4)} + \lambda_1^{(6)} - \cdots \qquad (91k)$$

$$Q_4 = T_4 - U_4 = \lambda_4^{(2)} - \lambda_4^{(4)} + \lambda_4^{(6)} - \cdots \qquad (91l)$$

Let us select a particular value of $c'$, say:

$$c' = c_0'$$

At the epoch corresponding to $c_0'$, $K_1 = 0°$ since we are at perihelion. Equation (91a) permits us to write Equation (90a) as:

$$Y = \kappa_1^{(0)} + \sum M_1 \begin{matrix}\cos\\ \sin\end{matrix}\, i c_0' \equiv (Y)$$

In this manner, $(\gamma)$ is defined as the value of $\gamma$ at $t = T_0$. Thus:

$$\kappa_1^{(0)} = (\gamma) - \sum M_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' \tag{92}$$

At $r = r'$, $K_1 = 90°$ and Equation (90a) by means of Equation (91g) becomes:

$$\gamma_{r'} = \kappa_1^{(0)} - \sum N_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0'$$

But at $r = r'$, $L_1 = 0°$ and Equations (90b) and (91e) give:

$$\gamma_{r'} = \lambda_1^{(0)} + \sum P_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0'$$

Since these expressions must be identical, we have:

$$\lambda_1^{(0)} + \sum P_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' = \kappa_1^{(0)} - \sum N_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0'$$

Combining this with Equation (92):

$$\lambda_1^{(0)} = (\gamma) - \sum (M_1 + N_1 + P_1) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' \tag{93}$$

We proceed in this manner at the other points of separation and obtain the following equations for the constants of integration.

$$\kappa_1^{(0)} = (\gamma) - \sum M_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' \tag{94a}$$

$$\lambda_1^{(0)} = (\gamma) - \sum (M_1 + N_1 + P_1) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' \tag{94b}$$

$$\kappa_2^{(0)} = (\gamma) - \sum (M_1 + N_1 + P_1 + Q_1 - N_2) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' \tag{94c}$$

$$\kappa_3^{(0)} = (\gamma) - \sum (M_4 + N_4 + P_4 + Q_4 - N_3) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} ic_0' \tag{94d}$$

$$\lambda_4^{(0)} = (\gamma) - \sum (M_4 + N_4 + P_4) \begin{matrix}\cos\\ \sin\end{matrix} ic_0 \tag{94e}$$

$$\kappa_4^{(0)} = (\gamma) - \sum M_4 \begin{matrix}\cos\\ \sin\end{matrix} ic_0' \tag{94f}$$

During the $x^{th}$ revolution after $T_0$, we have:

$$c' = c_x'$$

$$(\gamma)_x = \gamma \quad \text{at} \quad t = T_x$$

Then in the same manner as Equations (94) were obtained:

$$\kappa_1^{(0)} = (\gamma)_x - \sum M_1 \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{95a}$$

$$\lambda_1^{(0)} = (\gamma)_x - \sum (M_1 + N_1 + P_1) \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{95b}$$

$$\kappa_2^{(0)} = (\gamma)_x - \sum (M_1 + N_1 + P_1 + Q_1 - N_2) \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{95c}$$

$$\kappa_3^{(0)} = (\gamma)_x - \sum (M_4 + N_4 + P_4 + Q_4 - N_3) \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{95d}$$

$$\lambda_4^{(0)} = (\gamma)_x - \sum (M_4 + N_4 + P_4) \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{95e}$$

$$\kappa_4^{(0)} = (\gamma)_x - \sum (M_4) \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{95f}$$

We must now relate $(\gamma)$, $(\gamma)_1$, $\cdots$ $(\gamma)_x$. Consider the first aphelion passage of the comet after $t = T_0$. Using Equation (94c) for $\kappa_2^{(0)}$ in Equation (90c) gives:

$$\begin{aligned}\gamma &= (\gamma) - \sum (M_1 + N_1 + P_1 + Q_1 - N_2 - M_2) \begin{matrix}\cos\\ \sin\end{matrix} ic_0' \\ &= (\gamma) - 2\sum (R_1 + T_1 - R_2) \begin{matrix}\cos\\ \sin\end{matrix} ic_0'\end{aligned}$$

where we have recognized that at aphelion, $\kappa_2 = 180°$. But, at aphelion $\kappa_3 = 180°$ and we may consider this to be the beginning of the revolution with $c' = c_1'$. Equation (95d) gives $\kappa_3^{(0)}$ with $x = 1$ and using this value in Equation (90d), we have:

$$\gamma = (\gamma)_1 - 2\sum(R_4 + T_4 - R_3) \begin{matrix}\cos\\ \sin\end{matrix} ic_1'$$

Since these two expressions for $\gamma$ must be identical, we have:

$$(\gamma)_1 = (\gamma) - 2\sum(R_1 + T_1 - R_2) \begin{matrix}\cos\\ \sin\end{matrix} ic_0'$$

$$- 2\sum(R_4 + T_4 - R_3) \begin{matrix}\cos\\ \sin\end{matrix} ic_1'$$

In general, for any two consecutive values, $(\gamma)_{x-1}$ and $(\gamma)_x$, we have:

$$(\gamma)_x = (\gamma)_{x-1} - 2\sum(R_1 + T_1 - R_2) \begin{matrix}\cos\\ \sin\end{matrix} ic_{x-1}'$$

$$- 2\sum(R_4 + T_4 - R_3) \begin{matrix}\cos\\ \sin\end{matrix} ic_x' \tag{96}$$

We use Equations (96) to relate $(\gamma)_x$ to $(\gamma)$ and find:

$$(\gamma)_x = (\gamma) - 2\sum(R_1 + T_1 - R_2)\left[\begin{matrix}\cos\\ \sin\end{matrix} ic_0' + \begin{matrix}\cos\\ \sin\end{matrix} ic_1' + \cdots + \begin{matrix}\cos\\ \sin\end{matrix} ic_{x-1}'\right]$$

$$+ 2\sum(R_4 + T_4 - R_3)\left[\begin{matrix}\cos\\ \sin\end{matrix} ic_1' + \begin{matrix}\cos\\ \sin\end{matrix} ic_2' + \cdots + \begin{matrix}\cos\\ \sin\end{matrix} ic_x'\right] \tag{97}$$

Let $\Delta$ be the mean motion of the planet during a mean revolution of the comet. Then:

$$c_1' = c_0' + \Delta$$

$$c_2' - c_1' + \Delta = c_0' + 2\Delta$$

$$c_x' = c_0' + x\Delta \tag{98}$$

Two trigonometric identities are useful.

$$1 + \cos i\Delta + \cos 2i\Delta + \cdots + \cos (x-1)\, i\Delta = \frac{1}{2} + \frac{\sin\left(x - \frac{1}{2}\right) i\Delta}{2 \sin \frac{1}{2}\, i\Delta}$$

$$\sin i\Delta + \sin 2i\Delta + \cdots + \sin (x-1)\, i\Delta = \frac{1}{2} \cot \frac{1}{2}\, i\Delta - \frac{\cos\left(x - \frac{1}{2}\right) i\Delta}{2 \sin \frac{1}{2}\, i\Delta}$$

Multiplying the first identity by $\cos ic_0'$ and the second by $\sin ic_0'$ and subtracting, we have:

$$\cos ic_0' + \cos ic_1' + \cdots + \cos ic_{x-1}' = \frac{1}{2} \cos ic_0' - \frac{1}{2} \cot \frac{1}{2} \Delta \sin ic_0' + \frac{1}{2} \cot \frac{1}{2} \Delta \sin ic_x' - \frac{1}{2} \cos ic_x'$$

Multiplying the first identity by $\sin ic_0'$ and the second by $\cos ic_0'$ and adding:

$$\sin ic_0' + \sin ic_1' + \cdots + \sin ic_{x-1}' = \frac{1}{2} \sin ic_0' + \frac{1}{2} \cot \frac{i}{2} \Delta \cos ic_0' - \frac{1}{2} \cot \frac{i}{2} \Delta \cos ic_x' - \frac{1}{2} \sin ic_x'$$

These expressions are to be substituted in Equation (97). We must separate the sine and cosine terms. The quantities with subscript $c$ are to be used when $\gamma$ is given by a cosine series and the quantities with subscript $s$ are to be used when $\gamma$ is given by a sine series. Equation (97) becomes:

$$\begin{aligned}
(\gamma)_x = (\gamma) &- \sum \left(R_{1c} + T_{1c} - R_{2c} + R_{4c} + T_{4c} - R_{3c}\right) \cos ic_0' \\
&+ \sum \left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - T_{4c} + R_{3c}\right) \cot \frac{i}{2} \Delta \sin ic_0' \\
&- \sum \left(R_{1s} + T_{1s} - R_{2s} + R_{4s} + T_{4s} - R_{3s}\right) \sin ic_0' \\
&- \sum \left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} + R_{3s}\right) \cot \frac{i}{2} \Delta \cos ic_0' \\
&+ \sum \left(R_{1c} + T_{1c} - R_{2c} + R_{4c} + T_{4c} - R_{3c}\right) \cos ic_x' \\
&- \sum \left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - T_{4c} + R_{3c}\right) \cot \frac{i}{2} \Delta \sin ic_x' \\
&+ \sum \left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} + R_{3s}\right) \sin ic_x' \\
&+ \sum \left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} + R_{3s}\right) \cot \frac{i}{2} \Delta \cos ic_x' \qquad (99)
\end{aligned}$$

As $c_0'$ is a constant depending on the initial conditions, its effect is included in $(\gamma)$ which must be found from astronomical observations. We omit all terms dependent on $c_0'$. The subscript $x$ on $c'$ is dropped and $c'$ is the mean anomaly of the planet at the perihelion passage of the comet closest to the time at which we are determining the perturbed orbital elements.

Taking account of the above remarks and using Equation (99) for $(\gamma)_x$, Equations (95) become:

$$\kappa_1^{(0)}\zeta = (\gamma) + \sum\left(-S_{1c} + T_{1c} - R_{2c} + R_{4c} + T_{4c} - R_{3c} + X_s\right)\cos ic'$$

$$+ \sum\left(-S_{1s} + T_{1s} - R_{2s} + R_{4s} + T_{4s} - R_{3s} - X_c\right)\sin ic' \qquad \text{(100a)}$$

$$\lambda_1^{(0)} = (\gamma) + \sum\left(-R_{1c} - U_{1c} - R_{2c} + R_{4c} + T_{4c} - R_{3c} + X_s\right)\cos ic'$$

$$+ \sum\left(-R_{1s} - U_{1s} - R_{2s} + R_{4s} + T_{4s} - R_{3s} - X_c\right)\sin ic' \qquad \text{(100b)}$$

$$\kappa_2^{(0)} = (\gamma) + \sum\left(-R_{1c} - T_{1c} - S_{2c} + R_{4c} + T_{4c} - R_{3c} + X_s\right)\cos ic'$$

$$+ \sum\left(-R_{1s} - T_{1s} - S_{2s} + R_{4s} + T_{4s} - R_{3s} - X_c\right)\sin ic' \qquad \text{(100c)}$$

$$\kappa_4^{(0)} = (\gamma) + \sum\left(R_{1c} + T_{1c} - R_{2c} - S_{4c} + T_{4c} - R_{3c} + X_s\right)\cos ic'$$

$$+ \sum\left(R_{1s} + T_{1s} - R_{2s} - S_{4s} + T_{4s} - R_{3s} - X_c\right)\sin ic' \qquad \text{(100d)}$$

$$\lambda_4^{(0)} = (\gamma) + \sum\left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - U_{4c} - R_{3c} + X_s\right)\cos ic'$$

$$+ \sum\left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - U_{4s} - R_{3s} - X_c\right)\sin ic' \qquad \text{(100e)}$$

$$\kappa_4^{(0)} = (\gamma) + \sum\left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - T_{4c} - S_{3c} + X_s\right)\cos ic'$$

$$+ \sum\left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} - S_{3s} - X_c\right)\sin ic' \qquad \text{(100f)}$$

where:

$$X_c \equiv \left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - T_{4c} + R_{3c}\right) \cot \frac{i}{2} \Delta \tag{101a}$$

$$X_s \equiv \left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} + R_{3s}\right) \cot \frac{i}{2} \Delta \tag{101b}$$

In general then:

$$\Upsilon = (\Upsilon) + \sum \left(\kappa_1^{(0)} + \kappa_1^{(2)} \cos 2K + \cdots\right) \begin{matrix}\cos\\ \sin\end{matrix} ic' \tag{102}$$

If for some value of $i$, say $i = i_f$, we have:

$$i_f \Delta = m \cdot 360^\circ \tag{103}$$

where $m$ is an integer, $\cot (i/2)\Delta$ becomes infinite and the above expressions are useless. According to Equation (98) together with Equation (103):

$$c_x' = c_0 + \frac{mx}{i_f} \cdot 360^\circ$$

This may be written as:

$$i_f c_x' = i_f c_0' + mx \cdot 360^\circ$$

Since $mx = 0, 1, 2 \ldots$, we have:

$$\begin{matrix}\cos\\ \sin\end{matrix} i_f c_0' = \begin{matrix}\cos\\ \sin\end{matrix} i_f c_1' = \cdots = \begin{matrix}\cos\\ \sin\end{matrix} i_f c_x' \tag{104}$$

Equation (97) becomes:

$$\begin{aligned}(\Upsilon)_x = (\Upsilon) &- 2x\left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - T_{4c} + R_{3c}\right) \cos i_f c' \\ &- 2x\left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} + R_{3s}\right) \sin i_f c'\end{aligned} \tag{105}$$

Equations (95) become:

$$\kappa_1^{(0)} = (\gamma) - \left(R_{1c} + S_{1c}\right) \cos i_f c' - xz_c \cos i_f c'$$

$$- \left(R_{1s} + S_{1s}\right) \sin i_f c' - xz_s \sin i_f c' \tag{106a}$$

$$\lambda_1^{(0)} = (\gamma) - \left(2R_{1c} + T_{1c} + U_{1c}\right) \cos i_f c' - xz_c \cos i_f c'$$

$$- \left(2R_{1s} + T_{1s} + U_{1s}\right) \sin i_f c' - xz_s \sin i_f c' \tag{106b}$$

$$\kappa_2^{(0)} = (\gamma) - \left(2R_{1c} + 2T_{1c} - R_{2c} + S_{2c}\right) \cos i_f c' - xz_c \cos i_f c'$$

$$- \left(2R_{1s} + 2T_{1s} - R_{2s} + S_{2s}\right) \sin i_f c' - xz_s \sin i_f c' \tag{106c}$$

$$\kappa_3^{(0)} = (\gamma) - \left(2R_{4c} + 2T_{4c} - R_{3c} + S_{3c}\right) \cos i_f c' - xz_c \cos i_f c'$$

$$- \left(2R_{4s} + 2T_{4s} - R_{3s} + S_{3s}\right) \sin i_f c' - xz_s \sin i_f c' \tag{106d}$$

$$\lambda_4^{(0)} = (\gamma) - \left(2R_{4c} + T_{4c} + U_{4c}\right) \cos i_f c' - xz_c \cos i_f c'$$

$$- \left(2R_{4s} + T_{4s} + U_{4s}\right) \sin i_f c' - xz_s \sin i_f c' \tag{106e}$$

$$\kappa_4^{(0)} = (\gamma) - \left(R_{4c} + S_{4c}\right) \cos i_f c' - xz_c \cos i_f c'$$

$$- \left(R_{4s} + S_{4s}\right) \sin i_f c' - xz_s \sin i_f c' \tag{106f}$$

where:

$$z_c \equiv 2\left(R_{1c} + T_{1c} - R_{2c} - R_{4c} - T_{4c} + R_{3c}\right) \tag{107a}$$

$$z_s \equiv 2\left(R_{1s} + T_{1s} - R_{2s} - R_{4s} - T_{4s} + R_{3s}\right) \tag{107b}$$

This special case will always arise for $i_f = 0$.

In order to calculate the perturbations of the elliptic elements involving the mean motion, we must integrate again. If $\gamma$ represents the first integral obtained above, the integral, $\int \gamma \, dt$ is required. Let us call this integral $z$.

The general form of $\gamma$ is :

$$\gamma = (\gamma) + \sum \left(h^{(0)} + h^{(1)} \sin k + h^{(2)} \cos 2k + \cdots \right) \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} ic' \tag{108}$$

The sine terms may be absent in some cases as they are for our division of the ellipse into six segments. The form of dt is given by Equation (82) and hence $\gamma$ dt is of the form:

$$dk \sum \left(\phi_1 \cos k + \phi_2 \sin 2k + \cdots \right) \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} ic'$$

The integral is:

$$\text{const} + \sum \left(\phi_1 \sin k - \frac{1}{2} \phi_2 \cos 2k + \cdots \right) \begin{smallmatrix}\cos \\ \sin\end{smallmatrix} ic' \tag{109}$$

This is exactly the form of the first integral, Equation (89). The constants of integration can be found in the same manner as for the first integral. The special case, $i_f \Delta = 2\pi m$, requires separate consideration with regard to the terms multiplied by $x$. The terms independent of $x$ are dealt with as in the first integration. The terms in $x$ are of the form const $\times$ $x$, and dt is of the form of Equation (82). Since $x$ is independent of the partial anomaly, we will have the form:

$$z = \int \gamma\, dt = \text{const} + x\left[\beta_1 \cos 2K + \cdots \right]$$

where we are considering only the terms involving $x$ and, since in our case sine terms do not appear, only cosine terms are included. We will have an integral of this form for each segment of the ellipse. Thus:

$$z = j_1^{(0)} + x\left(j_1^{(2)} \cos 2K_1 + j_1^{(4)} \cos 4K_1 + \cdots \right) \tag{110a}$$

$$z = \ell_1^{(0)} + x\left(\ell_1^{(2)} \cos 2L_1 + \ell_1^{(4)} \cos 4L_1 + \cdots \right) \tag{110b}$$

$$z = j_2^{(0)} + x\left(j_2^{(2)} \cos 2K_2 \quad j_2^{(4)} \cos 4K_2 + \cdots \right) \tag{110c}$$

$$z = j_3^{(0)} + x\left(j_3^{(2)} \cos 2K_3 + j_3^{(4)} \cos 4K_3 + \cdots \right) \tag{110d}$$

$$z \quad \ell_4^{(0)} + x\left(\ell_4^{(2)} \cos 2L_4 + \ell_4^{(4)} \cos 4L_4 + \cdots \right) \tag{110e}$$

$$z \quad j_4^{(0)} + x\left(j_4^{(2)} \cos 2K_4 + j_4^{(4)} \cos 4K_4 + \cdots \right) \tag{110f}$$

As in Equations (91), define;

$$M_1' = R_1' + S_1' = j_1^{(2)} + j_1^{(4)} + \cdots ,$$

$$M_2' = R_2' + S_2' = j_2^{(2)} + j_2^{(4)} + \cdots , \qquad \text{etc.,}$$

$$N_1' = R_1' - S_1' = j_1^{(2)} - j_1^{(4)} + \cdots ,$$

$$N_2' = R_2' - S_2' = j_2^{(2)} - j_2^{(4)} + \cdots , \qquad \text{etc.} \tag{111}$$

Let $(z)_x$ be the value of $z$ at the $x^{th}$ passage of the perturbed body through its perihelion after the epoch. A calculation analogous to that which led to Equations (94) gives:

$$j_1^{(0)} = (z)_x - x M_1' \tag{112a}$$

$$\ell_1^{(0)} = (z)_x - x\left(M_1' + N_1' + P_1'\right) \tag{112b}$$

$$j_2^{(0)} = (z)_x - x\left(M_1' + N_1' + P_1' + Q_1' - N_2'\right) \tag{112c}$$

$$j_3^{(0)} = (z)_x - x\left(M_4' + N_4' + P_4' + Q_4' - N_3'\right) \tag{112d}$$

$$\ell_4^{(0)} = (z)_x - x\left(M_4' + N_4' + P_4'\right) \tag{112e}$$

$$j_4^{(0)} = (z)_x - x M_4' \tag{112f}$$

Also:

$$(z)_x = (z)_{x-1} - 2(x-1)\left(R_1' + T_1' - R_2'\right) + 2x\left(R_4' + T_4' - R_3'\right)$$

which gives:

$$(z)_x = (z) - 2\left[(x-1) + (x-2) + \cdots + 2 + 1\right]\left(R_1' + T_1' - R_2'\right) + 2\left[x + (x-1) + \cdots + 2 + 1\right]\left(R_4' + T_4' - R_3'\right) \tag{113}$$

But:

$$1 + 2 + \cdots + (x-1) = \frac{1}{2}x^2 - \frac{1}{2}x$$

and:

$$1 + 2 + \cdots + x = \frac{1}{2} x^2 + \frac{1}{2} x$$

Equation (113) becomes:

$$(z)_x = (z) - x^2 \left(R_1' + T_1' - R_2' - R_4' - T_4' + R_3'\right) + x\left(R_1' + T_1' - R_2' + R_4' + T_4' - R_3'\right) \tag{114}$$

Define:

$$w \equiv R_1' + T_1' - R_2' - R_4' - T_4' + R_3' \tag{115}$$

Equations (112) become:

$$j_1^{(0)} = (z) + x\left(- S_1' + T_1' - R_2' + R_4' + T_4' - R_3'\right) - x^2 w \tag{116a}$$

$$\ell_1^{(0)} = (z) + x\left(- R_1' - U_1' - R_2' + R_4' + T_4' - R_3'\right) - x^2 w \tag{116b}$$

$$j_2^{(0)} = (z) + x\left(- R_1' - T_1' - S_2' + R_4' + T_4' - R_3'\right) - x^2 w \tag{116c}$$

$$j_3^{(0)} = (z) + x\left(R_1' + T_1' - R_2' - R_4' - T_4' - S_3'\right) - x^2 w \tag{116d}$$

$$\ell_4^{(0)} = (z) + x\left(R_1' + T_1' - R_2' - R_4' - U_4' - R_3'\right) - x^2 w \tag{116e}$$

$$j_4^{(0)} = (z) + x\left(R_1' + T_1' - R_2' - S_4' + T_4' - R_3'\right) - x^2 w \tag{116f}$$

The determination of the constants of integration has been carried out for a division of the ellipse into six sections using the partial anomalies $K_1$, $K_2$, $K_3$, $K_4$, $L_1$, and $L_4$. It is of interest to determine what changes occur for different divisions of the ellipse and for the use of different anomalies.

If we use $L_2$ and $L_3$ rather than $L_1$ and $L_4$, we must replace:

$$P_1,\ Q_1,\ P_4,\ Q_4,\ T_1,\ T_4,\ U_1,\ U_4$$

by:

$$-Q_2,\ -P_2,\ -Q_3,\ -P_3,\ -T_2,\ -T_3,\ +U_2,\ +U_3\ ,$$

respectively.

If we used $L_1$ and $L_2$ instead of $L_1$ and $L_4$, we would make the above changes only for those constants relating to $L_4$. If we used more intermediate anomalies to divide the ellipse into more than six parts or more than two parts using these anomalies, we simply increase the number of P's, Q's, etc.

If we omit certain points of separation, we simply omit the corresponding constants unless we omit the point of separation at perihelion or aphelion. If we use the same partial anomaly on both sides of perihelion or aphelion, i.e., if we have no point of separation at perihelion or aphelion, the equation representing the element $\gamma$ will be of the form of Equation (108), that is, sine terms will be present. This will change the form of the equations.

Suppose, in our example, we omit the point of separation at perihelion. In all of the equations we will have $M_1 = M_4$. In addition, Equations (90a) and (90f) will combine and take the form:

$$\gamma = \kappa^{(0)} + \sum \left( \kappa^{(1)} \sin k + \kappa^{(2)} \cos 2k + \cdots \right) \begin{matrix} \cos \\ \sin \end{matrix} ic' \tag{117}$$

Define:

$$E \equiv G + H \equiv \kappa^{(1)} + \kappa^{(2)} - \kappa^{(3)} - \kappa^{(4)} + \cdots \tag{118a}$$

$$F \equiv G - H \equiv \kappa^{(1)} - \kappa^{(2)} - \kappa^{(3)} + \kappa^{(4)} + \kappa^{(5)} - \cdots \tag{118b}$$

Let us omit Equation (90f) and all quantities dependent on it. Equation (90a) in the form of Equation (117) describes this segment. We see that in all of the equations for the constants of integration, we drop all terms with subscript 4 and change:

$$\kappa_1^{(0)},\ M_1,\ N_1,\ R_1,\ S_1$$

to:

$$\kappa^{(0)},\ -E,\ -F,\ -G,\ -H$$

Now, omit the point of separation at aphelion. Equations (90c) and (90d) will combine and take the form:

$$\gamma = \kappa_1^{(0)} + \sum \left( \kappa_1^{(1)} \sin k_1 + \kappa_1^{(2)} \cos 2k_1 + \cdots \right) \begin{matrix} \cos \\ \sin \end{matrix} ic' \tag{119}$$

Define:

$$E_1 \equiv G_1 + H_1 \equiv \kappa_1^{(1)} + \kappa_1^{(2)} - \kappa_1^{(3)} - \kappa_1^{(4)} + \cdots \tag{120a}$$

$$F_1 \equiv G_1 - H_1 \equiv \kappa_1^{(1)} - \kappa_1^{(2)} - \kappa_1^{(3)} + \kappa_1^{(4)} + \kappa_1^{(5)} - \cdots \tag{120b}$$

Let us omit Equation (90d), and consequently, drop all terms with subscript 3. Then we must change: $\kappa_2^{(0)}$, $M_2$, $N_2$, $R_2$, $S_2$ to:

$$\kappa_1^{(0)}, \ -E_1, \ -F_1, \ -G_1, \ -H_1$$

In this case, we must take $x$ to be equal to the number of complete revolutions of the comet since the epoch if the time we are considering corresponds to the anomalies $K_1$, $L_1$, or $k_1$, and must take $x$ to be equal to this number plus one if the time we are considering pertains to $L_4$ or $K_4$.

We could also omit Equation (90c) and write Equation (90d) as:

$$Y = \kappa_{11}^{(0)} + \sum \left(\kappa_{11}^{(1)} \sin k_1 + \kappa_{11}^{(2)} \cos 2k_1 + \cdots\right) \begin{matrix}\cos \\ \sin\end{matrix} ic' . \tag{121}$$

Define:

$$E_{11} = G_{11} + H_{11} = \kappa_{11}^{(1)} + \kappa_{11}^{(2)} - \kappa_{11}^{(3)} - \kappa_{11}^{(4)} + \cdots \tag{122a}$$

$$F_{11} = G_{11} - H_{11} = \kappa_{11}^{(1)} - \kappa_{11}^{(2)} - \kappa_{11}^{(3)} + \kappa_{11}^{(4)} + \cdots \tag{122b}$$

consequently, we drop all terms with subscript 2 and change: $\kappa_3^{(0)}$, $M_3$, $N_3$, $R_3$, $S_3$ to:

$$\kappa_{11}^{(0)}, \ F_{11}, \ E_{11}, \ G_{11}, \ -H$$

In this case, $x$ has its usual definition when using $K_1$ and $L_1$, but must be increased by unity when using $k_1$, $L_4$, and $K_4$.

In the method of partial anomalies, the integrations have not led to the divisors $(in - i'n')$ and $(in - i'n')^2$ which appear in other methods. Instead, we obtain the multipliers $\cot (i/2)\Delta$ and $\cot^2 (i/2)\Delta$. In the case of near commensurability, these multipliers become large and lead to large perturbations of long period. When $in - i'n'$ is small, it may be that both $i$ and $i'$ have large values and, consequently, by ordinary methods we must develope many terms of the series in both $i$ and $i'$. In the method of mean anomalies, this is necessary for only one of the indices because of the rapid convergence of the series involving the partial anomalies.

## IV. THE PERTURBATION EQUATIONS

In the following, the variables $\gamma$, $\Xi$, $\Psi$, $\rho$, and $q$, first introduced by Hansen (Reference 3), wiii be used to designate the elliptic elements.

The perturbation equations are:

$$\frac{d\gamma}{dt} = \frac{2\,an}{\sqrt{1-e^2}}\left[\left(\frac{a}{r}+\frac{1}{1-e^2}\right)\cos f\,\frac{\partial\Omega}{\partial f}+\frac{e}{1-e^2}\frac{\partial\Omega}{\partial f}+\frac{a}{r}\sin f\cdot r\frac{\partial\Omega}{\partial r}\right] \tag{123a}$$

$$\frac{d\Psi}{dt} = \frac{2\,an}{\sqrt{1-e^2}}\left[\left(\frac{a}{r}+\frac{1}{1-e^2}\right)\sin f\,\frac{\partial\Omega}{\partial f}-\frac{a}{r}\cos f\cdot r\frac{\partial\Omega}{\partial r}\right] \tag{123b}$$

$$\frac{d\Xi}{dt} = -\frac{3\,an}{\sqrt{1-e^2}}\frac{\partial\Omega}{\partial f} \tag{123c}$$

$$\frac{dp}{dt} = -\frac{an}{\sqrt{1-e^2}}\cos i\sin I\sin f\sin(f'+\nu-k)\frac{\partial\Omega}{\partial H} \tag{123d}$$

$$\frac{dq}{dt} = -\frac{an}{\sqrt{1-e^2}}\cos i\sin I\cos f\sin(f'+\nu-k)\frac{\partial\Omega}{\partial H} \tag{123e}$$

In addition:

$$\frac{d\delta z}{dt} = \gamma\frac{r}{a}\cos f+\Psi\frac{r}{a}\sin f+\Xi \tag{124a}$$

$$\frac{dw}{dt} = \gamma\frac{n}{2\sqrt{1-e^2}}\sin f-\Psi\frac{n}{2\sqrt{1-e^2}}(\cos f+e) \tag{124b}$$

$$\delta s = \delta q\sin f-\delta p\cos f \tag{124c}$$

In these equations, $n\delta z$ is the perturbation in the mean anomaly or mean longitude, $w$ is the perturbation in the natural logarithm of the radius vector, and $\delta s$ is the perturbation in the sine of the latitude. The elements $\gamma$, $\Psi$, and $\Xi$ are obtained by integrating Equations (123a), (123b) and (123c), and $\delta p$ and $\delta q$ are the integrals of Equations (123d) and (123e). Also, $I$ designates the inclination of the orbit of the comet with respect to the orbit of the planet, $i$ is the inclination of the orbit of the comet with respect to the ecliptic, and $\delta s$ is measured with respect to the ecliptic. The quantity $\Omega$ is the perturbation function defined by:

$$\Omega \equiv m'\left[\frac{1}{\Delta}-\frac{r}{r'^2}H\right] \tag{125}$$

$m'$ and $r'$ being the mass and radius vector, respectively, of the perturbing body, H the cosine of the angle between the radius vector r and $r'$, and $\Delta$ is the distance between the planet and the comet. In addition, $\nu + k$ is the angle measured in the plane of the orbit of the comet, and measured from the ascending node of the orbit of the comet on the orbit of the planet to the perihelion of the comet, while $\nu - k$ is the angle measured in the plane of the orbit of the planet, and measured from the ascending node of the orbit of the comet on the orbit of the planet to the perihelion of the planet. Consequently:

$$H = \cos(f + \nu + k)\cos(f' + \nu - k) + \cos I \sin(f + \nu + k)\sin(f' + \nu - k) \tag{126}$$

Also:

$$\Delta^2 = r^2 + r'^2 - 2rr'H \tag{127}$$

We also have:

$$0 = 3r \cos f \, d\gamma + 3r \sin f \, d\Psi + 4a \, d\Xi \tag{128}$$

$$w = -\frac{1}{2}\gamma\frac{r}{a}\cos f - \frac{1}{2}\Psi\frac{r}{a}\sin f - \frac{2}{3}\Xi \tag{129}$$

write:

$$\Omega = \Omega_1 + \Omega_2$$

$$\Omega_1 = \frac{m'}{\Delta}$$

$$\Omega_2 = -m'\frac{r}{r'^2}H \tag{130}$$

Then:

$$\frac{\partial\Omega_1}{\partial f} = m'\frac{rr'}{\Delta^3}\frac{\partial H}{\partial f}$$

$$r\frac{\partial\Omega_1}{\partial r} = m'\frac{rr'}{\Delta^3}H - m'\frac{r^2}{\Delta^3}$$

$$\frac{\partial\Omega_1}{\partial H} = m'\frac{rr'}{\Delta^3} \tag{131}$$

By Equation (126):

$$\frac{dH}{df} = -\sin(f+\nu+k)\cos(f'+\nu-k) + \cos I\cos(f+\nu+k)\sin(f'+\nu-k) \tag{132}$$

Equations (131) become:

$$\frac{\partial\Omega_1}{\partial f} = -m'\frac{rr'}{\Delta^3}\left[\sin(f+\nu+k)\cos(f'+\nu-k) - \cos I\cos(f+\nu+k)\sin(f'+\nu-k)\right]$$

$$r\frac{\partial\Omega_1}{\partial r} = m'\frac{rr'}{\Delta^3}\left[\cos(f+\nu+k)\cos(f'+\nu-k) + \cos I\sin(f+\nu+k)\sin(f+\nu-k)\right] - m'\frac{r^2}{\Delta^3}$$

$$\frac{\partial\Omega_1}{\partial H} = m'\frac{rr'}{\Delta^3} \tag{133}$$

Define:

$$\cos(\nu-k) \equiv \gamma\sin G$$

$$\cos I\sin(\nu-k) \equiv \gamma\cos G$$

$$\nu + k + G \equiv \Gamma \tag{134a}$$

and:

$$-\sin(\nu-k) \equiv \gamma'\sin G'$$

$$\cos I\cos(\nu-k) \equiv \gamma'\cos G'$$

$$\nu + k + G' \equiv \Gamma' \tag{134b}$$

Let $u'$ be the eccentric anomaly of the planet. The elliptic equations give:

$$r'\cos f' = a'\cos u' - e'a'$$

$$r'\sin f' = a'\sqrt{1-e'^2}\sin u' \tag{135}$$

Equations (134), and (135) allow Equations (133) to be written as:

$$\frac{\partial\Omega_1}{\partial f} = m'\frac{a'r}{\Delta^3}\left[\begin{array}{c}\gamma\cos(f+\Gamma)\cos u' - e'\gamma\cos(f+\Gamma)\\ +\gamma'\sqrt{1-e'^2}\cos(f+\Gamma')\sin u'\end{array}\right] \tag{136a}$$

$$r\,\frac{\partial\Omega_1}{\partial r} = m'\,\frac{a'\,r}{\Delta^3}\left[\begin{array}{l}\gamma\sin(f+\Gamma)\cos u' - e'\,\gamma\sin(f+\Gamma) \\ +\gamma'\,\sqrt{1-e'^2}\,\sin(f+\Gamma')\sin u' - \dfrac{r}{a'}\end{array}\right] \tag{136b}$$

$$\frac{\partial\Omega_1}{\partial H} = m'\,\frac{r r'}{\Delta^3} \tag{136c}$$

Also:

$$r'\sin(f'+\nu-k) = -a'\,\gamma'\sin G'\cos u' + e'\,a'\,\gamma'\sin G' + a'\,\gamma\,\sqrt{1-e'^2}\,\sin G\sin u' \tag{137}$$

Equations (123) may then be written:

$$\frac{d\gamma}{dk} = m'\,\frac{r^3}{\Delta^3}\left[2P\cos u' + 2Q\sin u' + R\right] \tag{138a}$$

$$\frac{d\Psi}{dk} = m'\,\frac{r^3}{\Delta^3}\left[2P_1\cos u' + 2Q_1\sin u' + R_1\right] \tag{138b}$$

$$\frac{d\Xi}{dk} = m'\,\frac{r^3}{\Delta^3}\left[2P_2\cos u' + 2Q_2\sin u' + R_2\right] \tag{138c}$$

$$\frac{dp}{dk} = m'\,\frac{r^3}{\Delta^3}\left[2P_3\cos u' + 2Q_3\sin u' + R_3\right]\cos i \tag{138d}$$

$$\frac{dq}{dk} = m'\,\frac{r^3}{\Delta^3}\left[2P_4\cos u' - 2Q_4\sin u' + R_4\right]\cos i \tag{138e}$$

where:

$$P = \frac{a'\,a^2\gamma}{r\sqrt{1-e^2}}\cdot\frac{n\,dt}{r^2\,dk}\left[\cos\Gamma + \frac{r(\cos f + e)}{a\left(1-e^2\right)}\cos(f+\Gamma)\right] \tag{139a}$$

$$Q = \frac{a'\,a^2\gamma\sqrt{1-e'^2}}{r\sqrt{1-e^2}}\cdot\frac{n\,dt}{r^2\,dk}\left[\cos\Gamma' + \frac{r(\cos f + e)}{a\left(1-e^2\right)}\cos(f+\Gamma')\right] \tag{139b}$$

$$R = -\frac{2a'\,a^2\gamma}{r\sqrt{1-e^2}}\cdot\frac{n\,dt}{r^2\,dk}\left[e\left(\cos\Gamma + \frac{r(\cos f + e)}{a\left(1-e^2\right)}\cos(f+\Gamma)\right) + \frac{r\sin f}{a'\,\gamma}\right] \tag{139c}$$

$$P_1 = -\frac{a'\,a^2\gamma}{r\sqrt{1-e^2}}\cdot\frac{n\,dt}{r^2\,dk}\left[\sin\Gamma - \frac{r\sin f}{a\left(1-e^2\right)}\cos(f+\Gamma)\right] \tag{139d}$$

$$Q_1 = -\frac{a' a^2 \gamma' \sqrt{1-e'^2}}{r \sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \left[ \sin\Gamma' - \frac{r \sin f}{a\left(1-e^2\right)} \cos(f + \Gamma') \right] \tag{139e}$$

$$R_1 = \frac{2a' a^2 \gamma}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \left[ e\left( \sin\Gamma - \frac{r \sin f}{a\left(1-e^2\right)} \cos(f+\Gamma) \right) + \frac{r \cos f}{a' \gamma} \right] \tag{139f}$$

$$P_2 = -\frac{3a' a \gamma}{1\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \cos(f + \Gamma) \tag{139g}$$

$$Q_2 = -\frac{3a' a\gamma' \sqrt{1-e'^2}}{2\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \cos(f + \Gamma') \tag{139h}$$

$$R_2 = \frac{3a' a\gamma e'}{\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \cos(f+\Gamma) \tag{139i}$$

$$P_3 = \frac{a' a\gamma' \sin G' \sin I}{2\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \sin f \tag{139j}$$

$$Q_3 = -\frac{a' a\gamma \sqrt{1-e'^2} \sin G \sin I}{2\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \sin f \tag{139k}$$

$$R_3 = -\frac{e' a' a\gamma' \sin G' \sin I}{\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \sin f \tag{139l}$$

$$P_4 = \frac{a' a\gamma' \sin G' \sin I}{2\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \cos f \tag{139m}$$

$$Q_4 = -\frac{a' a\gamma \sqrt{1-e'^2} \sin G \sin I}{2\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \cos f \tag{139n}$$

$$R_4 = -\frac{e' a' a\gamma' \sin G' \sin I}{\sqrt{1-e^2}} \cdot \frac{ndt}{r^2 dk} \cos f \tag{139o}$$

Expressions for ndt in terms of the various partial anomalies were given in section I.

We must now take account of $\Omega_2$ given by Equation (130). Recall:

$$\Omega_2 = -m' \frac{r}{r'^2} H$$

Then:

$$\frac{\partial \Omega_2}{\partial f} = -\frac{m' \, r}{r'^2} \frac{\partial H}{\partial f} = -\frac{m' \, r r'}{r'^3} \frac{\partial H}{\partial f} \tag{140a}$$

$$r \frac{\partial \Omega_2}{\partial r} = -\frac{m' \, r}{r'^2} H = -\frac{m' \, r r'}{r'^3} H \tag{140b}$$

$$\frac{\partial \Omega_2}{\partial H} = -m' \frac{r}{r'^2} = -m' \frac{r r'}{r'^3} \tag{140c}$$

If, in Equations (131), we change $\Delta^3$ to $-r'^3$ and omit the term $-m' \, r^2/\Delta^3$, these equations are identical to Equations (140). The results for the second part of the perturbing function are the same as for the first part if in Equations (139c) and (139f) we omit the term $r \sin f/a' \gamma$ and $r \cos f/a' \gamma$, respectively, and if, in all equations, we replace $\Delta^3$ by $-r'^3$. Consequently, Equations (138) and (139) show that the expressions for $d\Psi/dk$, $d\gamma/dk$, etc., arising from the second part of the perturbing function, contain terms of the form:

$$\frac{\cos u' - e'}{r'^3} \quad \text{and} \quad \frac{\sin u'}{r'^3}$$

These expressions may be developed in a Fourier series in $\cos iu'$ or $\sin iu'$. The coefficients of these series will be given by:

$$\mu_i' = \frac{1}{a'^3 \pi} \int_0^{\pi} \frac{(\cos u' - \sin \phi') \cos iu' \, du'}{(1 - \sin \phi' \cos u')^3} \tag{141a}$$

$$\nu_i' = \frac{1}{a'^3 \pi} \int_0^{\pi} \frac{\sin u' \sin iu' \, du'}{(1 - \sin \phi' \cos u')^3} \tag{141b}$$

where:

$$e' = \sin \phi' \tag{142}$$

and the elliptic equation relating $r'$ to $a'$, $e'$, and $u'$ has been taken into account. Integration by parts gives:

$$\frac{1}{\pi} \int_0^{\pi} \frac{\cos iu' \, du'}{1 - \sin \phi' \cos u'} = \frac{\tan^i \frac{1}{2} \phi'}{\cos \phi'} \tag{143a}$$

Differentiating this with respect to $\phi'$:

$$\frac{\cos\phi'}{\pi}\int_0^{\pi}\frac{\cos u' \cos iu' \, du'}{(1-\sin\phi'\cos u')^2} = \frac{i \tan^{i-1}\frac{1}{2}\phi'}{2\cos^2\frac{1}{2}\phi' \cos\phi'} + \frac{\tan^i\frac{1}{2}\phi' \sin\phi'}{\cos^2\phi'} \tag{143b}$$

But, multiplying Equation (143a) by:

$$\frac{1-\sin\phi'\cos u'}{1-\sin\phi'\cos u'}$$

gives:

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos iu' \, du'}{(1-\sin\phi'\cos u')^2} - \frac{\sin\phi'}{\pi}\int_0^{\pi}\frac{\cos u' \cos iu' \, du'}{(1-\sin\phi'\cos u')^2} = \frac{\tan^i\frac{1}{2}\phi'}{\cos\phi'} \tag{143c}$$

Equations (143b) and (143c) give:

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos iu' \, du'}{(1-\sin\phi'\cos u')^2} = \frac{\tan^i\frac{1}{2}\phi'}{\cos^3\phi'}(1+i\cos\phi') \tag{143d}$$

Applying this procedure a second time:

$$\frac{1}{\pi}\int_0^{\pi}\frac{\cos iu' \, du'}{(1-\sin\phi'\cos u')^3} = \frac{\tan^i\frac{1}{2}\phi'}{2\cos^5\phi'}\left[2+\sin^2\phi' + 3i\cos\phi' + i^2\cos^2\phi'\right] \tag{144}$$

Applying the above results to Equations (141), we have:

$$\mu_i' = \frac{\tan^{i-1}\frac{1}{2}\phi'}{4\cos^2\frac{1}{2}\phi'\cos^3\phi'}\left[\sin^2\phi' + i\cos\phi' + i^2\cos^2\phi'\right] \tag{145a}$$

$$\nu_i' = \frac{\tan^{i-1}\frac{1}{2}\phi'}{4\cos^2\frac{1}{2}\phi'\cos^3\phi'}\left[i + i^2\cos\phi'\right] \tag{145b}$$

Consequently, the perturbation equations for the second part of the perturbation function, may be written as:

$$\frac{d\gamma}{dk} = r^3 P\mu_0 + r^3 P\mu_1 \cos u' + r^3 P\mu_2 \cos 2u' + \cdots$$

$$+ r^3 Q_1 \nu_1 \sin u' + r^3 Q \nu_2 \sin 2u' + \cdots \tag{146a}$$

$$\frac{d\Psi}{dk} = r^3 P_1 \mu_0 + r^3 P_1 \mu_1 \cos u' + r^3 P_1 \mu_2 \cos 2u' + \cdots$$

$$+ r^3 Q_1 \nu_1 \sin u' + r^3 Q_1 \nu_2 \sin 2u' + \cdots \tag{146b}$$

$$\frac{d\Xi}{dk} = r^3 P_2 \mu_0 + r^3 P_2 \mu_1 \cos u' + r^3 P_2 \mu_2 \cos 2u' + \cdots$$

$$+ r^3 Q_2 \nu_1 \sin u' + r^3 Q_2 \nu_2 \sin 2u' + \cdots \tag{146c}$$

$$\frac{dp}{dk} = \begin{bmatrix} r^3 P_3 \mu_0 + r^3 P_3 \mu_1 \cos u' + r^3 P_3 \mu_2 \cos 2u' + \cdots \\ + r^3 Q_3 \nu_1 \sin u' + r^3 Q_3 \nu_2 \sin 2u' + \cdots \end{bmatrix} \cos i \tag{146d}$$

$$\frac{dq}{dk} = \begin{bmatrix} r^3 P_4 \mu_0 + r^3 P_4 \mu_1 \cos u' + r^3 P_4 \mu_2 \cos 2u' + \cdots \\ + r^3 Q_4 \nu_1 \sin u' + r^3 Q_4 \nu_2 \sin 2u' + \cdots \end{bmatrix} \cos i \tag{146e}$$

where:

$$\mu_i = -4 \frac{m'}{a'^3} \mu_i'$$

$$\nu_i = -4 \frac{m'}{a'^3} \nu_i' \tag{147}$$

except:

$$\mu_0 = -\frac{m'}{a'^3} \frac{\sin \phi'}{\cos^3 \phi'}$$

The quantities $P$, $Q$, $P_1$, $Q_1$, etc., are given by Equations (139). The total value of $d\gamma/dk$, $d\Psi/dk$, etc., is given by the sum of Equations (138) and (146).

The series given by Equations (146) may not converge when $r > a$ unless only a very small segment of the ellipse near aphelion is represented by the superior anomaly. This results since $r^3$ is a factor and if it is large, the convergence is weak at best. To overcome this difficulty, we divide the perturbation function as follows:

$$\Omega = \frac{m'}{\Delta} - m' \left[\frac{1}{r} + \frac{r'}{r^2} H\right] + m' \left[\frac{1}{r} + \frac{r'}{r^2} H - \frac{r}{r'^2} H\right] \tag{148}$$

The first term has already been considered. The third term may be integrated in finite terms and will be considered later.

Consider the second term:

$$\Omega = -m' \left[\frac{1}{r} + \frac{r'}{r^2} H\right]$$

Then:

$$\frac{\partial \Omega}{\partial f} = -m' \frac{r'}{r^2} \frac{\partial H}{\partial f} = -m' \frac{r r'}{r^3} \frac{\partial H}{\partial f} \qquad (148a)$$

$$r \frac{\partial \Omega}{\partial r} = 2m' \frac{r'}{r^2} H + \frac{m'}{r} = 2m' \frac{r r'}{r^3} H + m' \frac{r^2}{r^3} \qquad (148b)$$

$$\frac{\partial \Omega}{\partial H} = -m' \frac{r'}{r^2} = -m' \frac{r r'}{r^3} \qquad (148c)$$

Equations (148) will be identical with Equations (131) if, in Equations (131), $\Delta^3$ is replaced by $-r^3$, and $-3m' r r'/r^2$ is added to the second Equation (131). Proceeding as we did in deriving Equations (138), we find:

$$\begin{aligned}\frac{d\Upsilon}{dk} = &- \left[m' R + e' \frac{6m' a' a^2 \gamma}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2\, dk} \sin f \sin (f + \Gamma)\right] \\ &- \left[2m' P - \frac{6m' a' a^2 \gamma}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2\, dk} \sin f \sin (f + \Gamma)\right] \cos u' \\ &- \left[2m' Q - \frac{6m' a' a^2 \gamma' \sqrt{1-e'^2}}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2\, dk} \sin f \sin (f + \Gamma')\right] \sin u'\end{aligned} \qquad (149a)$$

$$\begin{aligned}\frac{d\Psi}{dk} = &- \left[m' R_1 - e' \frac{6m' a' a^2 \gamma}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2\, dk} \cos f \sin (f + \Gamma)\right] \\ &- \left[2m' P_1 + \frac{6m' a' a^2 \gamma}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2\, dk} \cos f \sin (f + \Gamma)\right] \cos u' \\ &- \left[2m' Q_1 + \frac{6m' a' a^2 \gamma' \sqrt{1-e'^2}}{r\sqrt{1-e^2}} \cdot \frac{ndt}{r^2\, dk} \cos f \sin (f + \Gamma')\right] \sin u'\end{aligned} \qquad (149b)$$

$$\frac{d\Xi}{dk} = -\left[m' R_2 + 2m' P_2 \cos u' + 2m' Q_2 \sin u'\right] \tag{149c}$$

$$\frac{dp}{dk} = -\left[m' R_3 + 2m' P_3 \cos u' + 2m' Q_3 \sin u'\right] \cos i \tag{149d}$$

$$\frac{dq}{dk} = -\left[m' R_4 + 2m' P_4 \cos u' + 2m' Q_4 \sin u'\right] \cos i \tag{149e}$$

Equations (149) will be very convergent when the superior anomaly is used but will not be convergent with the inferior anomaly. Consequently, Equations (146) are used with the inferior anomaly and Equations (149) with the superior anomaly.

Consider the third term of Equation (148)

$$\Omega_3 = m' \left[\frac{1}{r} + \frac{r'}{r^2} H - \frac{r}{r'^2} H\right]$$

Proceeding as in Equations (131), (132), (133), (134), (135), (136), and (137), the perturbation equations for $\Omega_3$ are:

$$\frac{d\gamma}{dt} = 2m' \frac{an}{\sqrt{1 - e^2}} \left[\gamma \frac{ar'}{r^3} \cos f \cos (f + \Gamma) \cos f'\right.$$

$$+ \gamma \frac{\cos f + e}{1 - e^2} \cos (f + \Gamma) \cos f' \left(\frac{r'}{r^2} - \frac{r}{r'^2}\right)$$

$$- 2\gamma \frac{ar'}{r^3} \sin f \sin (f + \Gamma) \cos f' - \gamma \frac{a}{r'^2} \cos \Gamma \cos f'$$

$$+ \gamma' \frac{\cos f + e}{1 - e^2} \cos (f + \Gamma') \sin f' \left(\frac{r'}{r^2} - \frac{r}{r'^2}\right)$$

$$+ \gamma' \frac{ar'}{r^3} \cos f \cos (f + \Gamma') \sin f' - \gamma' \frac{a}{r'^2} \cos \Gamma' \sin f'$$

$$\left.- 2\gamma' \frac{ar'}{r^3} \sin f \sin (f + \Gamma') \sin f' - \frac{a}{r^2} \sin f\right] \tag{150a}$$

$$\frac{d\Psi}{dt} = 2m' \frac{an}{\sqrt{1-e^2}} \left[ \gamma \frac{ar'}{r^3} \sin f \cos(f+\Gamma) \cos f' \right.$$

$$+ \gamma \frac{\sin f}{1-e^2} \cos(f+\Gamma) \cos f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right)$$

$$+ 2\gamma \frac{ar'}{r^3} \cos f \sin(f+\Gamma) \cos f' + \gamma \frac{a}{r'^2} \sin\Gamma \cos f'$$

$$+ \gamma' \frac{\sin f}{1-e^2} \cos(f+\Gamma') \sin f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right)$$

$$+ \gamma' \frac{ar'}{r^3} \sin f \cos(f+\Gamma') \sin f' + \gamma' \frac{a}{r'^2} \sin\Gamma' \sin f'$$

$$\left. + 2\gamma' \frac{ar'}{r^3} \cos f \sin(f+\Gamma') \sin f' + \frac{a}{r^2} \cos f \right] \tag{150b}$$

$$\frac{d\Xi}{dt} = -3m' \frac{an}{\sqrt{1-e^2}} \left[ \gamma \cos(f+\Gamma) \cos f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right) \right.$$

$$\left. + \gamma' \cos(f+\Gamma') \sin f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right) \right] \tag{150c}$$

$$\frac{dp}{dt} = m' \frac{an}{\sqrt{1-e^2}} \cos i \sin I \left[ \gamma' \sin G' \sin f \cos f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right) \right.$$

$$\left. - \gamma \sin G \sin f \sin f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right) \right] \tag{150d}$$

$$\frac{dq}{dt} = m' \frac{an}{\sqrt{1-e^2}} \cos i \sin I \left[ \gamma' \sin G' \cos f \cos f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right) \right.$$

$$\left. - \gamma \sin G \cos f \sin f' \left( \frac{r'}{r^2} - \frac{r}{r'^2} \right) \right] \tag{150e}$$

The perturbation equations have now been written in a form proper for integration in all cases.

## V. SERIES EXPANSION OF $\Delta^{-n}$

Consider a general periodic function.

$$X = \alpha_0 + \alpha_1 \cos x + \alpha_2 \cos^2 x + \cdots + \alpha_n \cos^n x$$
$$+ \sin x \left(\beta_0 + \beta_1 \cos x + \beta_2 \cos^2 x + \cdots + \beta_{n-1} \cos^{n-1} x\right) \tag{151}$$

We will show that this can always be written in the form:

$$X = c\left[1 - q_1 \cos\left(x - Q_1\right)\right]\left[1 - q_2 \cos\left(x - Q_2\right)\right] \cdots \left[1 - q_n \cos\left(x - Q_n\right)\right] \tag{152}$$

Define:

$$\tan \frac{1}{2} x \equiv y \tag{153}$$

Then:

$$\sin x = \frac{2y}{1 + y^2} \qquad \cos x = \frac{1 - y^2}{1 + y^2} \tag{154}$$

Equation (151) may be written as:

$$X = \frac{1}{\left(1 + y^2\right)^n}\Big[\alpha_0\left(1 + y^2\right)^n + \alpha_1\left(1 - y^2\right)\left(1 + y^2\right)^{n-1}$$
$$+ \alpha_2\left(1 - y^2\right)^2\left(1 + y^2\right)^{n-2} + \cdots + \alpha_n\left(1 - y^2\right)^n\Big]$$
$$+ \frac{2y}{\left(1 + y^2\right)^n}\Big[\beta_0\left(1 + y^2\right)^{n-1} + \beta_1\left(1 - y^2\right)\left(1 + y^2\right)^{n-2}$$
$$+ \cdots + \beta_{n-1}\left(1 - y^2\right)^{n-1}\Big] \tag{155}$$

Suppose:

$$X\left(1 + y^2\right)^n = 0 \tag{156}$$

Equation (156) is of degree $2n$ and may be written:

$$0 = c_{2n}\, y^{2n} + c_{2n-1}\, y^{2n-1} + \cdots + c_1 y + c_0 \tag{157}$$

Let the roots be:

$$a_1, a_2, \cdots, a_n$$

$$b_1, b_2, \cdots, b_n .$$

Then:

$$X\left(1+y^2\right)^n = c_{2n}\left(y-a_1\right)\left(y-b_1\right)\left(y-a_2\right)\left(y-b_2\right) \cdots \left(y-a_n\right)\left(y-b_n\right) \tag{158}$$

or:

$$X\left(1+y^2\right)^n = c_{2n}\left(y^2-s_1 y+p_1\right)\left(y^2-s_2 y+p_2\right) \cdots \left(y^2-s_n y+p_n\right) \tag{159}$$

where:

$$\begin{array}{lllll} a_1+b_1 & = s_1 & \qquad & a_1 b_1 & = p_1 \\ a_1+b_2 & = s_2 & & a_2 b_2 & = p_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_n+b_n & = s_n & & a_n b_n & = p_n \end{array} \tag{160}$$

We can always select $s_i$ and $p_i$ to be real. Since:

$$y = \tan \frac{1}{2} x$$

$$1+y^2 = \sec^2 \frac{1}{2} x$$

Consequently:

$$\begin{aligned} X = {} & c_{2n} \cos^{2n} \frac{1}{2} x \left(\tan^2 \frac{1}{2} x - s_1 \tan \frac{1}{2} x + p_1\right) \\ & \times \left(\tan^2 \frac{1}{2} x - s_2 \tan \frac{1}{2} x + p_2\right) \cdots \left(\tan^2 \frac{1}{2} x - s_n \tan \frac{1}{2} x + p_n\right) \end{aligned} \tag{161}$$

If we multiply each factor by $\cos^2 1/2\, x$, it will take the form:

$$\sin^2 \frac{1}{2} x - s \sin \frac{1}{2} x \cos \frac{1}{2} x + p \cos^2 \frac{1}{2} x$$

or:

$$\frac{1}{2}(1+p) - \frac{1}{2}(1-p)\cos x - \frac{1}{2} s \sin x$$

Define:

$$\begin{array}{lll}
\dfrac{1-p_1}{1+p_1} \equiv q_1 \cos Q_1 & & \dfrac{s_1}{1+p_1} \equiv q_1 \sin Q_1 \\
\dfrac{1-p_2}{1+p_2} \equiv q_2 \cos Q_2 & & \dfrac{s_2}{1+p_2} \equiv q_2 \sin Q_2 \\
\vdots \qquad\quad \vdots & & \vdots \\
\dfrac{1-p_n}{1+p_n} \equiv q_n \cos Q_n & & \dfrac{s_n}{1+p_n} \equiv q_n \sin Q_n
\end{array} \tag{162}$$

and:

$$c \equiv \frac{1}{2^n} c_{2n} (1+p_1)(1+p_2) \cdots (1+p_n)$$

or

$$c \equiv \frac{c_0 (1+p_1)(1+p_2)(1+p_3) \cdots (1+p_n)}{2^n a_1 a_2 a_3 \cdots a_n b_1 b_2 b_3 \cdots b_n} \tag{163}$$

Equation (155) becomes:

$$X = c\left[1 - q_1 \cos(x - Q_1)\right] \cdots \left[1 - q_n \cos(x - Q_n)\right]$$

that is, we have proved Equation (152).

The case $n = 2$ will of some interest. Comparing terms in Equations (155) and (157), we have:

$$(\alpha_0 - \alpha_1 + \alpha_2) y^4 + 2(\beta_0 - \beta_1) y^3 + 2(\alpha_0 - \alpha_2) y^2 + 2(\beta_0 + \beta_1) y + (\alpha_0 + \alpha_1 + \alpha_2) = 0. \tag{164}$$

Equation (151) is:

$$X = \alpha_0 - \alpha_1 \cos x + \alpha_2 \cos^2 x - \beta_0 \sin x + \beta_1 \sin x \cos x \tag{165}$$

Write:

$$X = c\left[1 - \frac{q}{c}\cos(x - Q)\right]\left[1 - \frac{q_1}{c}\cos\left(x - Q_1\right)\right]$$

Comparing terms, we have:

$$\begin{aligned}
\alpha_0 &= c + \frac{qq_1}{c}\sin Q \sin Q_1 \\
\alpha_1 &= q\cos Q + q_1 \cos Q_1 \\
\alpha_2 &= \frac{qq_1}{c}\cos\left(Q + Q_1\right) \\
\beta_0 &= q\sin Q + q_1 \sin Q_1 \\
\beta_1 &= \frac{qq_1}{c}\sin\left(Q + Q_1\right)
\end{aligned} \tag{166}$$

Hence:

$$\tan\left(Q + Q_1\right) = \frac{\beta_1}{\alpha_2} \tag{167}$$

Define:

$$\begin{aligned}
Q + Q_1 &\equiv 2\theta \\
Q - Q_1 &\equiv 2\omega
\end{aligned} \tag{168}$$

Then:

$$\begin{aligned}
\tan 2\theta &= \frac{\beta_1}{\alpha_2} \\
Q &= \theta + \omega \\
Q_1 &= \theta - \omega
\end{aligned} \tag{169}$$

Eliminating $Q$ and $Q_1$ from Equations (166), we have:

$$\alpha_0 = c + \frac{qq_1}{c}\left(\sin^2\theta - \sin^2\omega\right)$$

$$\alpha_2 = \frac{qq_1}{c}\cos 2\theta$$

$$\alpha_1 = \left(q + q_1\right)\cos\theta\cos\omega - \left(q - q_1\right)\sin\theta\sin\omega$$

$$\beta_0 = \left(q + q_1\right)\sin\theta\cos\omega + \left(q - q_1\right)\cos\theta\sin\omega$$

$$\beta_1 = \frac{qq_1}{c}\sin 2\theta \tag{170}$$

Define:

$$\alpha_0' = \alpha_0 + \alpha_2\sin^2\theta - \beta_1\sin\theta\cos\theta$$

$$\alpha_1' = \alpha_1\cos\theta + \beta_0\sin\theta$$

$$\alpha_2' = \alpha_2\cos 2\theta + \beta_1\sin 2\theta$$

$$\beta_0' = \beta_0\cos\theta - \alpha_1\sin\theta \tag{171}$$

The quantity $\theta$ is determined by Equation (169). With the aid of Equations (171), Equations (170) become:

$$\alpha_0' = c - \frac{qq_1}{c}\sin^2\omega$$

$$\alpha_1' = \left(q + q_1\right)\cos\omega$$

$$\alpha_2' = \frac{qq_1}{c}$$

$$\beta_0' = \left(q - q_1\right)\sin\omega \tag{172}$$

We are going to require that the q's always be positive. In this particular case, this requires that $\alpha_2'$ be positive. Consequently, we select the value of $\theta$ satisfying Equation (169) so that $\alpha_2'$ is positive. Define:

$$q \cos\omega \equiv \alpha_1' - \eta \qquad q_1 \cos\omega \equiv \eta$$

$$q \sin\omega \equiv \beta_0' + \xi \qquad q_1 \sin\omega \equiv \xi$$

$$c \equiv \alpha_0' + \zeta \qquad \text{or} \qquad \zeta \equiv \frac{qq_1}{c} \sin^2\omega \tag{173}$$

Equations (172) and (173) give:

$$\left(\alpha_0' + \zeta\right) \zeta = \left(\beta_0' + \xi\right) \xi \tag{174a}$$

$$\alpha_2' \left(\alpha_0' + \zeta\right) = \left(\alpha_1' - \eta\right) \eta + \left(\beta_0' + \xi\right) \xi \tag{174b}$$

$$\left(\alpha_1' - \eta\right) \xi = \left(\beta_0' + \xi\right) \eta \tag{174c}$$

Equations (174) must be solved to obtain $\zeta$, $\eta$, and $\xi$. Equations (174a) and (174b) give, by subtraction:

$$\left(\alpha_0' + \zeta\right) \left(\alpha_2' - \zeta\right) = \left(\alpha_1' - \eta\right) \eta \tag{175}$$

Equation (174a) may be written:

$$\frac{1}{4} \beta_0'^{\,2} + \left(\alpha_0' + \zeta\right) \zeta = \left(\frac{1}{2} \beta_0' + \xi\right)^2 \tag{176}$$

Using Equation (174), we have:

$$\frac{1}{2} \beta_0' + \xi = \frac{1}{2} \alpha_1' \frac{\xi}{\eta} \tag{177}$$

Equations (176) and (177) yield:

$$\frac{1}{4} \beta_0^2 + \left(\alpha_0 + \zeta\right) \zeta = \frac{1}{4} \alpha_0'^{\,2} \frac{\xi^2}{\eta^2} \tag{178}$$

Now, Equations (174a) and (175) may be written as:

$$\beta_0 + \xi = \frac{\left(\alpha_0' + \zeta\right) \zeta}{\xi}$$

$$\alpha_1' - \eta = \frac{\left(\alpha_0' + \zeta\right) \left(\alpha_2' - \zeta\right)}{\eta}$$

Inserting these into Equation (174c) gives:

$$\frac{\xi^2}{\eta^2} = \frac{\zeta}{\alpha_2' - \zeta} \tag{179}$$

Eliminating $\xi^2/\eta^2$ between Equations (178) and (179) gives:

$$\zeta^3 + \left(\alpha_0' - \alpha_2'\right)\zeta^2 + \left(\frac{1}{4}\alpha_1'^2 + \frac{1}{4}\beta_0'^2 - \alpha_0'\alpha_2'\right)\zeta - \frac{1}{4}\beta_0'^2\alpha_2' = 0 \tag{180}$$

Equation (180) determines the possible values of $\zeta$. Equations (174a) and (175) may be written in the following form.

$$\xi^2 + \beta_0'\xi - \left(\alpha_0' + \zeta\right)\zeta = 0 \tag{181}$$

$$\eta^2 - \alpha_1'\eta + \left(\alpha_0' + \zeta\right)\left(\alpha_2' - \zeta\right) = 0 \tag{182}$$

For each value of $\zeta$, we can determine values of $\xi$ and $\eta$. To determine which roots of Equations (181) and (182) are to be taken together, the following condition must be satisfied.

$$\frac{\alpha_1' - \eta}{\eta} = \frac{\beta_0' + \xi}{\xi} \tag{183}$$

Which pair of roots satisfying Equation (183) is taken is unimportant as this simply results in a permutation of the factors in X.

We may have some use of the particular case, $\beta_1 = 0$, and $\alpha_2$ very small. If $\beta_1 = 0$, Equation (169) gives $\theta = 0$. Equations (171) become:

$$\alpha_0' = \alpha_0$$

$$\alpha_1' = \alpha_1$$

$$\alpha_2' = \alpha_2$$

$$\beta_0' = \beta_0$$

Also, from Equation (169):

$$\omega = Q = -Q_1$$

Define:

$$\beta_0 = f \sin F$$

$$\alpha_1 = f \cos F$$

Equations (180), (181), and (182) become:

$$\zeta = \alpha_2 \sin^2 F + \frac{4\alpha_0 \alpha_2}{f^2} \zeta - \frac{4(\alpha_0 - \alpha_2)}{f^2} \zeta^2 - \frac{4}{f^2} \zeta^3$$

$$\xi = \frac{(\alpha_0 + \zeta) \zeta}{\beta_0} - \frac{1}{\beta_0} \xi^2$$

$$\eta = \frac{(\alpha_0 + \zeta)(\alpha_0 - \zeta)}{\alpha_1} + \frac{1}{\alpha_i} \eta^2$$

If $\alpha_2$ is small, these equations may be quickly solved by iteration. Let us now return to the general case.

Recall Equation (155).

$$X = c \left[1 - q_1 \cos(x - Q_1)\right] \cdots \left[1 - q_n \cos(x - Q_n)\right]$$

Let:

$$f_i \equiv c - q_i \cos \phi_i$$

$$\phi_i \equiv x - Q_i \tag{184}$$

In problems of interest to us, $q_i < 1$. Define:

$$\phi_i \equiv 180^\circ - 2\phi_i'$$

Then:

$$f_i = (c + q_i) \left[1 - \frac{2q_i}{c + q_i} \sin^2 \phi_i'\right] \tag{185}$$

We may write:

$$X = c \cdot f_1 \cdot f_2 \cdots f_n$$

We will be required to determine $X^{-n/2}$ where $n$ is an integer. Now:

$$X^{-n/2} = c^{-n/2} f_1^{-n/2} \cdot f_2^{-n/2} \cdots f_n^{-n/2}$$

But, $f_i$ is given by Equation (185) and this is of the form of Equation (27a) which we have developed in detail for $n = 1$. In the general case, the equations of section II become:

$$\left(1 - \epsilon^2 \sin^2 k\right)^{-n/2} = \alpha_0 - 2\alpha_2 \cos 2k + \cdots$$

$$\alpha_{2i} = \pm \frac{2}{\pi} \int_0^{\pi/2} \frac{\cos 2ik\, dk}{\left(1 - \epsilon^2 \sin^2 k\right)^{n/2}}$$

$$0 = (2i + n - 2)\, \epsilon^2 \alpha_{2i-2} - 4i\left(2 - \epsilon^2\right) \alpha_{2i} + (2i - n + 2)\, \epsilon^2 \alpha_{2i+2}$$

$$p_{2i} \equiv \frac{2i + n - 2}{4i} \cdot \frac{\epsilon^2}{2 - \epsilon^2}\, \gamma_{2i} \equiv \frac{\alpha_{2i}}{\alpha_{2i-2}}$$

$$\lambda_{2i} \equiv \frac{(2i - n)(2i + n - 2)}{16i(i - 1)} \left(\frac{\epsilon^2}{2 - \epsilon^2}\right)^2$$

The approximation for $\gamma_{2i}$ is:

$$\gamma_{2i} = \sec^2 \frac{1}{2} X \left[1 - \frac{n(n - 2)}{4i(i + 1)} \sin^2 \frac{1}{2} X\right]$$

where:

$$\epsilon = \sin \psi$$

$$\cos \psi = \tan\left(45 - \frac{1}{2} X\right)$$

We will be somewhat interested in the case $n = 3$. We have:

$$\alpha_0^{(3)} = \frac{1}{\pi} \int_0^{\pi} \frac{dk}{\left(1 - \epsilon^2 \sin^2 k\right)^{3/2}}$$

Since:

$$\alpha_0^{(1)} = \frac{1}{\pi} \int_0^{\pi} \frac{dk}{\left(1 - \epsilon^2 \sin^2 k\right)^{1/2}}$$

and:

$$\alpha_2^{(3)} = \frac{1}{\pi} \int_0^{\pi} \frac{\cos 2k\, dk}{\left(1 - \epsilon^2 \sin^2 k\right)^{3/2}}$$

we have:

$$\alpha_0^{(1)} = \left(1 - \frac{1}{2}\epsilon^2\right)\alpha_0^{(3)} + \frac{1}{2}\epsilon^2 \alpha_2^{(3)}$$

But, the definition of $p_{2i}$ gives:

$$\alpha_2^{(3)} = p_2^{(3)} \alpha_0^{(3)}$$

and

$$\alpha_0^{(3)} = \frac{\alpha_0^{(1)}}{1 - \frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^2 p_2^{(3)}}$$

Returning to the expansion of $f^{-n/2}$, if we define:

$$\sin X \equiv \frac{q}{c} \tag{186}$$

we write:

$$f^{-n/2} = \mu_0 + 2\mu_1 \cos\phi + 2\mu_2 \cos 2\phi + \cdots \tag{187}$$

where:

$$\mu_i = \mu_0 \cdot p_2 \cdot p_4 \cdots p_{2i} \tag{188}$$

For $n = 1$:

$$\mu_0^{(1)} = \frac{(c + q_i)^{-1/2}}{\pi} \int_0^{\pi} \frac{dk}{\left(1 - \frac{2q_i}{c + q_i} \sin^2\phi_i'\right)^{1/2}} \tag{189}$$

For $n = 3$:

$$\mu_0^{(3)} = \frac{(c + q_i)^{-3/2}}{\pi} \int_0^{\pi} \frac{dk}{\left(1 - \frac{2q_i}{c + q_i} \sin^2\phi_i'\right)^{3/2}} \tag{190}$$

Hence:

$$\mu_0^{(3)} = \frac{\mu_0^{(1)}}{c - q_i\, p_2^{(3)}} \tag{191}$$

We now write:

$$\left[1 - q_1 \cos\left(x - Q_1\right)\right]^{-n/2} = \mu_0^{(1)} + 2\mu_1^{(1)} \cos Q_1 \cos x + 2\mu_2^{(1)} \cos 2Q_1 \cos 2x + \cdots$$

$$+ 2\mu_1^{(1)} \sin Q_1 \sin x + 2\mu_2^{(1)} \sin 2Q_1 \sin 2x + \cdots$$

$$\left[1 - q_2 \cos\left(x - Q_2\right)\right]^{-n/2} = \mu_0^{(2)} + 2\mu_1^{(2)} \cos Q_2 \cos x + 2\mu_2^{(2)} \cos 2Q_2 \cos 2x + \cdots$$

$$+ 2\mu_1^{(2)} \sin Q_2 \sin x + 2\mu_2^{(2)} \sin 2Q_2 \sin 2x + \cdots \text{ etc.}, \tag{192}$$

write:

$$X^{-n/2} = [m, 0]_c + 2[m, 1]_c \cos x + 2[m, 2]_c \cos 2x + \cdots$$

$$+ 2[m, 1]_s \sin x + 2[m, 2]_s \sin 2x + \cdots \tag{193}$$

where, $m$ is the number of factors in $X$. Then:

$$[m, \ell]_c = c^{-n/2} \sum \mu_{i'}^{(1)} \cdot \mu_{i''}^{(2)} \cdots \mu_{i^{(m)}}^{(m)} \cos\left(i' Q_1 + i'' Q_2 + \cdots + i^{(m)} Q_m\right) \tag{194}$$

and:

$$[m, \ell]_s = c^{-n/2} \sum \mu_{i'}^{(1)} \cdot \mu_{i''}^{(2)} \cdots \mu_{i^{(m)}}^{(m)} \sin\left(i' Q_1 + i'' Q_2 + \cdots + i^{(m)} Q_m\right) \tag{195}$$

where:

$$\ell = i' + i'' + \cdots + i^{(m)}$$

The sum is over all values of combinations of $i'$, $i''$, $\cdots$, $i^{(m)}$, which sum to $\ell$.

Recall the elliptic equations.

$$r' = a'(1 - e' \cos u')$$

$$r' \cos f' = a'(\cos u' - e')$$

$$r' \sin f' = a' \sqrt{1 - e'^2} \sin u'$$

Equations (126) and (127) were:

$$H = \cos(f + \nu + k)\cos(f' + \nu - k) + \cos I \sin(f + \nu + k)\sin(f' + \nu - k)$$

and:

$$\Delta^2 = r^2 + r'^2 - 2rr'H$$

Combining these with Equations (134) for $\gamma$, G, $\Gamma$, and $\gamma'$, G', $\Gamma'$, we have:

$$r'H = Aa' \cos u' + Ba' \sin u' - Ae'a' \qquad (196)$$

where:

$$A \equiv \gamma \sin(f + \Gamma)$$

$$B \equiv \gamma' \sqrt{1 - e'^2} \sin(f + \Gamma') \qquad (197)$$

and:

$$\Delta^2 = a'^2 + r^2 + 2e'Aa'r - 2\left(Aa'r + a'^2 e'\right)\cos u' - 2Ba'r \sin u' + a'^2 e'^2 \cos^2 u' \qquad (198)$$

We have now expressed $\Delta^2$ as a periodic function of $u'$. By means of the equations of elliptic motion, we may convert this to an infinite series in powers of sines and cosines of the mean anomaly. If, in this series, we neglect powers higher than the $n$th, we will have $\Delta^2$ as a periodic function of $\sin g'$ and $\cos g'$ of the $n$th degree and the results of this section may be applied. The actual application will be discussed in Section VI.

## VI. A DISCUSSION OF THE APPLICATION OF THE METHOD OF PARTIAL ANOMALIES

Sections I-V of this report provide most of the theory of the method of partial anomalies necessary to be able to apply it to problems of interest. This section will summarize the earlier results and fill in the few gaps left in the previous sections.

The method of partial anomalies as developed by Hansen (Reference 1) applies directly to the perturbations produced by a body, considered as a point mass, on the unperturbed two body elliptic motion. The generalization to more than one perturbing body is obvious, since the perturbations may be considered as additive in all cases likely to occur. An important application of this method may be to high eccentricity artificial Earth satellites as well as to interplanetary probes. In fact, this method will converge for any value of $e$, the eccentricity. The only difficulty occurs if the perturbation is so large that the orbital character of the motion is completely changed so as to reverse the direction of motion. Needless to say, this causes difficulty in any known method.

Consequently, in order to apply the method to artificial satellites, the method must be adapted to the perturbing force due to the departure of the Earth's shape from sphericity. The author of this report is currently beginning an attack on this problem in order to determine the quantitative value of this method as applied to high eccentricity satellites and probes. Let us now return to the problem at hand.

Hansen was interested in applying this method to comets, most of which have high eccentricities, and partially determined the perturbations on Encke's comet produced by the Earth. In what follows, I shall refer to the comet and the Earth, rather than the perturbed body and the perturbing body, respectively. It is clear that this "specialization" does not cause a loss of generality while it may maintain Hansen's line of thought in a clearer fashion.

The problem may be divided into three major parts. First, the definition of the inferior and superior anomalies and their auxiliaries, $\kappa_1$, $\kappa_2$, $\kappa_3$, $\kappa_4$, and the intermediate anomalies. The equations of elliptic motion are then expressed in terms of the appropriate partial anomaly. This part of the problem is dealt with in Section I. We have seen that the comet's ellipse may be divided into any number of segments in an arbitrary manner. Exactly how this is done will be determined by the specific problem and by the choice between rapid convergence and the number of series which must be developed. Since a new set of series is required for each segment of the ellipse, it is obvious that we will reach the point where the work required by $n$ sets of series is not worth the greater convergence obtained by dividing the ellipse into $n$ segments rather than $n - 1$ segments.

Once we have expressed the motion of the comet in terms of the partial anomalies, we turn to the second part of the problem. This is the development of various functions, such as the perturbing function, in trigonometric series of multiples of sines and cosines of the partial anomalies. Once expressed in this form, integrations are readily performed. This development in trigonometric series has been discussed in Sections II, IV, and V.

The expressions for $\sin f$ and $ndt$ in terms of the partial anomalies in Section I contain radicals such as $\left(1 - \epsilon^2 \sin^2 k\right)^{-1/2}$ In Section II, these radicals are developed into trigonometric series convenient for integration. The end result of Section II is that the elliptic motion of the comet has been expressed in terms of the partial anomalies in a readily integrable form.

In Section IV, the perturbation equations are developed into trigonometric series involving the eccentric anomaly of the Earth. The sines and cosines of this eccentric anomaly may be converted to sines and cosines of the mean anomaly of the Earth by means of the elliptic equations and is generally performed by means of Bessel functions. This is discussed in standard texts on celestial mechanics. Finally, by means of Equation (84):

$$g' = n' t + c'$$

we express the perturbation equations in terms of sines and cosines of $c'$. Since $c'$ is independent of the partial anomalies, integrations are readily performed.

In Section V, $\Delta^{-n}$ is developed into trigonometric series involving the Earth's eccentric anomaly. Again, these may be converted to a trigonometric series in $c'$ and again we have a form convenient for integration.

The second part of the problem having been completed, we turn to the third and final problem. This is the determination of the constants of integration, which was discussed in Section III. Since each segment of the ellipse is represented by a different set of series, each segment will have its own constants of integration. These are determined by initial conditions, such as the initial values of the orbital elements of the comet, and by the condition that we have continuity at the points of separation which divide the various segments of the ellipse.

## REFERENCES


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